Math 160A - Fall 2021 - Quiz 13 - Upload by 12:00pm (Noon), Sunday, December 5

Start Time:

Your name: Answer Key

Stop Time:

Integrity signature:

Time limit 15 minutes, not counting download and upload. Please add explanation if over 17 minutes.

There are two pages, the second with "cheat" sheet material. Only the first page needs to be turned in.

1. Prove that there exists an FO proof of

$$\forall x\,(P(x) \to Q(x)) \to \exists x\, P(x) \to \exists x\, Q(x).$$

Answer #1

By Deduction Theorem, it suffices to prove

$\forall x\,(P(x) \to Q(x)) \vdash \exists x\, P(x) \to \exists x\, Q(x)$

This is the same as $\forall x\,(P(x) \to Q(x)) \vdash \neg\forall x\,\neg P(x) \to \neg\forall x\,\neg Q(x)$

By Tautology, suffices to show

$\forall x\,(P(x) \to Q(x)) \vdash \forall x\,\neg Q(x) \to \forall x\,\neg P(x)$.

By Deduction Theorem, suffices to show

$\forall x\,(P(x) \to Q(x)), \forall x\,\neg Q(x) \vdash \forall x\,\neg P(x)$

Here is the proof the that:

$\forall x\,(P(x) \to Q(x)), \forall x\,\neg Q(x) \vdash \forall x\,\neg Q(x)$ — Hyp

$\forall x\,(P(x) \to Q(x)), \forall x\,\neg Q(x) \vdash \neg Q(x)$ — UI rule

$\forall x\,(P(x) \to Q(x)), \forall x\,\neg Q(x) \vdash P(x) \to Q(x)$ — Hyp & UI rule

$\forall x\,(P(x) \to Q(x)), \forall x\,\neg Q(x) \vdash \neg P(x)$ — Modus Tollens

$\forall x\,(P(x) \to Q(x)), \forall x\,\neg Q(x) \vdash \forall x\,\neg P(x)$

Answer #2: From 1st line above, by Deduction Theorem, it suffices to show

$\forall x\,(P(x) \to Q(x)), \exists x\, P(x) \vdash \exists x\, Q(x)$

By Theorem on constants, it suffices to show

$\forall x\,(P(x) \to Q(x)), P(c) \vdash \exists x\, Q(x)$ Since c is "new".

This proof is:

$\forall x\,(P(x) \to Q(x)), P(c) \vdash \forall x\,(P(x) \to Q(x))$ — Hyp

$\forall x\,(P(x) \to Q(x)), P(c) \vdash P(c) \to Q(c)$ — UI rule

$\forall x\,(P(x) \to Q(x)), P(c) \vdash Q(c)$ — Hyp & M.P.

$\forall x\,(P(x) \to Q(x)), P(c) \vdash \forall x\,\neg Q(x) \to \neg Q(c)$ — UI axiom

$\forall x\,(P(x) \to Q(x)), P(c) \vdash \neg\forall x\,\neg Q(x)$ — Modus Tollen

and $\neg\forall x\,\neg Q(x)$ is the same as $\exists x\, Q(x)$.

Axioms and Inference Rules for FO-proofs

PL1: $A \to B \to A$

PL2: $(A \to B \to C) \to (A \to B) \to (A \to C)$

PL3: $\neg A \to A \to B$

PL4: $(\neg A \to A) \to A$

EQ1: $x = x$

EQ2: $x = y \to y = x$

EQ3: $x = y \to y = z \to x = z$

EQ$_f$: $y_1 = z_1 \to \cdots \to y_k = z_k \to f(y_1, \ldots, y_k) = f(z_1, \ldots, z_k)$

EQ$_P$: $y_1 = z_1 \to \cdots \to y_k = z_k \to P(y_1, \ldots, y_k) \to P(z_1, \ldots, z_k)$

Universal Instantiation (UI): $\forall x\, A(x) \to A(t)$.

Modus Ponens: $\dfrac{A \qquad A \to B}{B}$.

Generalization (Gen): $\dfrac{C \to A}{C \to \forall x\, A}$ $\quad$ (x not free in C)

$A \vee B$, $A \wedge B$, and $\exists x\, A$ abbreviate $\neg A \to B$, $\neg(A \to \neg B)$ and $\neg \forall x \neg A$.

Constructing FO-proofs

Derived rules:

Propositional rules: Modus Tollens, Hypothetical Syllogism and more generally Tautological Implication (TAUT).

Generalization: $\dfrac{A}{\forall x\, A}$

If t is substitutable for x in A:

UI Rule: $\dfrac{\forall x\, A(x)}{A(t)}$ $\qquad$ **Substitution:** $\dfrac{A(x)}{A(t)}$

For Deduction/Contradiction/By Cases, assume A is a sentence.

Deduction Theorem: $\Gamma \vdash A \to B$ iff $\Gamma, A \vdash B$.

Proof by Contradiction:

$\Gamma \vdash A$ iff $\Gamma \cup \{\neg A\}$ is inconsistent.

$\Gamma \vdash \neg A$ iff $\Gamma \cup \{A\}$ is inconsistent.

Proof by Cases: If $\Gamma, A \vdash B$ and $\Gamma, \neg A \vdash B$, then $\Gamma \vdash B$.

Theorem on Constants. c is a new constant symbol, not in Γ, A, B.

For $A(x)$ a formula, $\Gamma \vdash \forall x\, A(x)$ iff $\Gamma \vdash A(c)$.

If $\exists x\, A(x)$ is a sentence, $\Gamma \cup \{\exists x\, A(x)\} \vdash B$ iff $\Gamma \cup \{A(c)\} \vdash B$