$\Gamma \vDash A$ and $\Pi \supseteq \Gamma \Rightarrow \Pi \vDash A$

$\Gamma \vdash A$ and $\Pi \supseteq \Gamma \Rightarrow \Pi \vdash A$

$\{B : \Gamma \vdash B\}$ or $\{B : \Gamma \vDash B\}$

Consequences of Γ ; Theorems of Γ

Γ is a theory if $\Gamma = \{A : \Gamma \vDash A\}$

$Cn\,\Gamma$

$\neg\neg A \vdash A \iff \{\neg\neg A, \neg A\}$ is inconsistent

$\{\neg, \vee\}$ - adequate

$\{\neg, \wedge\}$ - also adequate

$\{\neg, \wedge, \vee\}$ - adequate

$\{\neg, \rightarrow\}$ - adequate

$\{\rightarrow, \oplus\}$ - adequate??

$p \oplus p$ $\quad \neg(p \rightarrow p)$

So far

p	$p \rightarrow p$	$p \oplus p$	$p \oplus \bot$	$p \oplus (p \oplus p)$
T	T	F	T	T
F	T	F	F	F

$\top$

$\bot$

p

$\neg p$?

p	$p \rightarrow (p \oplus p)$		$\neg p$	$p \oplus \top$	$p \oplus (p \rightarrow p)$
T	F	F	F	F	F
F	T	F	T	T	T

Yes it is adequate!

$\{\neg, \oplus\}$ is not adequate

p	$\neg p$	$p \oplus p$	$\neg(p \oplus p)$
T	F	F	T
F	T	F	T

Inconclusive

Question: Can we express $\vee$ (or $\wedge$, or $\rightarrow$) with $\neg, \oplus$

p	q	$p \oplus q$	$\neg(p \oplus q)$	$\neg p$	$\neg q$	$(p \oplus q \oplus (\neg q))$
T	T	F	T	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	T
F	F	F	T	T	T	T

$(\neg p \oplus q) \vDash\Dash \neg(p \oplus q)$

$p \oplus p \vDash\Dash \bot$

$\bot \oplus p \vDash\Dash \neg p$

$p \oplus q \vDash\Dash q \oplus p$

$p \oplus (q \oplus r) \vDash\Dash (p \oplus q) \oplus r$

$\oplus$ - Sum mod 2

$T \cup (A \vee B)$ is inconsistent $\Longleftrightarrow$ $T \cup \{A\}$ and $T \cup \{B\}$ are inconsistent.

$\Longrightarrow$ (Easy direction)

Know $A \vdash A \vee B$ $\qquad \vdash A \to A \vee B$ proved in class

Know $B \vdash A \vee B$ $\qquad B \to A \vee B$ is a PL-axiom

$B \to \neg A \to B$

Let C be any formula.

Want to show $T, A \vdash C$ and $T, B \vdash C$

By $A \vdash A \vee B$, $B \vdash A \vee B$, and by the assumption that $T, A \vee B \vdash C$ ($T \cup \{A \vee B\}$ is inconsistent)

we get $T, A \vdash C$ and $T, B \vdash C$.

$T \vdash A \to A \vee B$ and $T \vdash A \vee B \to C$

By Hypothetical Syllogism, $T \vdash A \to C$

So $T, A \vdash C$.

$\boxed{\text{Assume } \Gamma, A \text{ inconsistent and } \Gamma, B \text{ inconsistent.}}$

Want to show $\Gamma \cup \{A \vee B\}$ is inconsistent.

Let C be arbitrary. Want to $\Gamma, A \vee B \vdash C$

Use proof-by-cases:

<u>Show</u> $\Gamma, A \vee B, A \vdash C$

and <u>Show</u> $\Gamma, A \vee B, \neg A \vdash C$

i.e. $\Gamma, \neg A \to B, \neg A \vdash C$

$\Gamma, \neg A \to B, \neg A \vdash B$ by M.P.

and $\Gamma \vdash B \to C$ since $\Gamma \cup \{B\}$ is inconsistent

So $\Gamma, \neg A \to B, \neg A \vdash C$ MP.

$\Gamma, A \vdash C$ since $\Gamma \cup \{A\}$ is inconsistent

Thus $\Gamma, A \vee B, A \vdash C$

Proof by cases

$T \vdash A$ iff $T, B \vdash A$ and $T, \neg B \vdash A$

i.e. $T \vdash A$ iff $T \vdash B \to A$ and $T \vdash \neg B \to A$

Two cases B and $\neg B$

Generalized Proof by Cases

Suppose $T \vdash D \vee E$

Then $T \vdash A$ iff $T \vdash D \to A$ and $T \vdash E \to A$

Proof omitted

~~$T \cup \{\neg A\}$ iff $T \cup$~~ is ~~inconsistent~~