Completeness Theorem: $T$ is a set of sentences, $A$ is a formula:

(a) If $T \vdash A$, then $T \cup \{A\}$.
(b) If $T$ is consistent, then $T$ is satisfiable.

We prove part (b).

Languages: $T, A$ are $L$-sentences / $L$-formulae.

$L^+ = L \cup \{d_1, d_2, d_3, \ldots \}$ new constant symbols.

**Defn** A closed term is a term with no variables.

**Defn** A set $T$ is Henkin if for all sentences $\exists x A(x)$,

if $T \vdash \exists x A(x)$, then $T \vdash A(t)$ for some closed term $t$ "witness" for $\exists x A(x)$.

$\exists x A(x)$ means $\neg \forall x \neg A(x)$.

Alternatively, $T$ is Henkin if whenever $T \vdash \forall x \neg A(x)$,

then $T \vdash \neg A(t)$ for some closed term $t$.

**Defn**: $T$ is strongly Henkin if for all closed sentences $\forall x A(x)$,

$T \vdash A(c) \to \forall x A(x)$

for some constant symbol $c$. 
Lemma: If $\forall x A(x)$ is a sentence and $c$ is a "new" constant symbol (not in $\Gamma$), then $\Gamma \cup \{ A(c) \rightarrow \forall x A(x) \}$ is consistent.

Pf: Suppose this fails, $\Gamma \vdash (A(c) \rightarrow \forall x A(x))$ by Contradiction Principle. By That: $\Gamma \vdash A(c)$ and $\Gamma \vdash \forall x A(x)$.

By $\Gamma \vdash A(c)$, we have $\Gamma \vdash \forall x A(x)$, by Theorem on Constants. Contradicts consistency of $\Gamma$.

Theorem: If $\Gamma$ is a consistent set of $L$-sentences, then there exists a consistent, strongly Henkin set of $L^+$-sentences, called $\Delta$, such that $\Delta \supseteq \Gamma$.

Pf: Enumerate all $L^+$-sentences $\forall x_i \exists y_j A_j(x_i)$, $j = 1, 2, 3, \ldots$ so that no $d_j$ appears in $\forall x_i \exists y_k A_k(x_i, y_k)$ for $k < j$.

Let $\Gamma_0 = \Gamma$,

$\Gamma_{i+1} = \Gamma_i \cup \{ (A(d_j) \rightarrow \forall x_i \exists y_j A_j(x_i)) \}$

By induction, each $\Gamma_i$, is consistent. Set $\Delta = \bigcup \Gamma_i$.
Lindenbaum Theorem: If $\Delta$ is consistent and strongly Henkin, there is a consistent, complete and strongly Henkin $\mathcal{F} \models \Delta$ of $\mathcal{L}^+$-sentences.

Defn: $\mathcal{F}$ is complete if for all $\mathcal{L}^+$-sentences $A$, either $A \in \mathcal{F}$ or $\neg A \in \mathcal{F}$.

Proof: $D_0 = \Delta$, $D_i+1 = \{D_i \cup \{A_i\} \text{ if consistent} \}
\{D_i \cup \{\neg A_i\} \text{ otherwise.} \}

where $A_i$'s enumerate all $\mathcal{L}^+$-sentences.

Properties of $\mathcal{F}$: $A, B$ - $\mathcal{L}^+$-sentences

$A \in \mathcal{F}$ if $\neg A \notin \mathcal{F}$.

$A \rightarrow B \in \mathcal{F}$ if $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

If $\mathcal{F} + A$, then $A \in \mathcal{F}$.

$\neg \forall x A(x) \in \mathcal{F}$ iff for some closed term $t$) $\neg A(t) \in \mathcal{F}$.

$\mathcal{F}$ has a $A(d) \leftrightarrow \forall x A(x)$.
Structure 1: Symbol $\mathcal{L}$ in language $L$.

Define $\alpha$ by $|\alpha| =$ set of closed $\mathcal{L}^+$-terms.

$C_\alpha = c$

$P_\alpha = \{ \left< t_1, \ldots, t_k \right> : P(t_1, \ldots, t_k) \in \Pi \}$

$f_\alpha(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$

Claim: $\alpha \not\models \Pi$, hence $\alpha \not\vdash \Pi$.

$\alpha$ is a $\mathcal{L}^+$-structure.

Definition of truth, gives values to more complex formulas.

$\alpha \models \neg P(t_i) \iff \alpha \not\models P(t_i) \iff P(t_i) \not\models \Pi \iff \neg P(t_i) \models \Pi$.

$\alpha \not\models A \rightarrow B \iff \alpha \not\models A \land \alpha \models \neg B \iff A \not\vdash \Pi \land B \not\vdash \Pi \iff A \not\models B \not\vdash \Pi$. 

$\forall x \exists y \phi \iff \exists x \forall y \phi$ 

$\exists \phi$ 

$\forall \phi$
$\forall t \in T \forall x B(x) \iff \forall t \models B(t)$ for all $L^t$-terms $t$.

$\sigma(t) = t$ for all object assignments $\sigma$ and closed terms $t$.

$\forall t \models B(t)$ for the "Henk.- witness" $\forall x B(x)$

$\Leftrightarrow B(t) \in \Pi$ by induction hypothesis.

$\forall x B(x) \in \Pi$ since $B(t) \models \forall x B(x) \in \Pi$.

$\forall x B(x) \in \Pi \Rightarrow$ for all closed terms $\forall t \models B(t) \in \Pi$

$\Leftrightarrow \forall t \models B(t)$ for all closed terms.

by induction hypothesis.

$\Rightarrow \forall t \models B(t)[\sigma]$ for all object assignments $\sigma$.

$(\sigma/t)$ for arbitrary $t$.

$\Rightarrow \forall t \not\models \forall x B(x)$ by defn of truth.

So, $\forall t \not\models A \in \Pi$.

So, $\forall t \not\models A$ for all $A \in \Pi$.

$\sigma$ is a $L^t$-structure.
Not Hintikka: Let $\forall \exists \in \{\exists x P(x)\}$

$L = \{c, ?\}$

$\vdash \exists x (P(x) \rightarrow \forall y P(y))$ - logically valid

$\exists x (P(x) \rightarrow \forall y P(y)) \vdash (\forall x P(x) \rightarrow \forall y P(y))$
What is \( s \in \mathcal{L} \) the language?

Modify the \( \lambda \varepsilon \)

Define: \( s \in \varepsilon \) if \( s + \in \mathcal{T} \).

\( \sim \) is an equivalence relation.

\[ \llbracket s \rrbracket = \{ t : t \sim s \} = \text{the equivalence class that contains } s \]

Define: \( \mathcal{A} = \{ \llbracket s \rrbracket : s \text{ is a closed } L^+ \text{ term} \} \)

\[ c^a = \llbracket c \rrbracket \]

\[ f^a (\llbracket t_1 \rrbracket, \ldots, \llbracket t_k \rrbracket) = \llbracket f(t_1, \ldots, t_k) \rrbracket \]

\[ p^a (\llbracket t_1 \rrbracket, \ldots, \llbracket t_k \rrbracket) = \text{True if } P(t_1, \ldots, t_k) \in \mathcal{T} \]

Lemma: \( f, P \) are well-defined.

Theorem: \( \varnothing \notin \mathcal{B} \) if \( \mathcal{B} \in \mathcal{T} \) for all \( L^+ \)-sentences \( B \)

Q.E.D. Completeness Theorem.
Compactness Theorem: Let $\mathcal{T}$ be a set of sentences.

1. $\mathcal{T} \models A$ if and only if $\mathcal{T}' \models A$ for some finite $\mathcal{T}' \subseteq \mathcal{T}$.

2. $\mathcal{T}$ is unsatisfiable iff some finite subset of $\mathcal{T}$ is unsatisfiable.

Pf: An immediate corollary.
Application: Fix \( L \).

Theorem: There is no sentence \( B \) such that \( A \vdash A \).

Proof: We'll show no sentence \( C \) such that \( A \vdash C \)

\( A \equiv C \) if and only if \( |a| \) is finite.

Let \( T = \{ \text{Atleast}_1, \text{Atleast}_2, \text{Atleast}_3, \ldots \} \).

\( A \models T \) iff \( |a| \) is finite.

If \( C \) had the above property, \( T \not\models C \).

So some finite \( T' \subseteq T \), has \( T' \models C \).

But this means \( \text{Atleast}_k \models C \) for some finite \( k \).

Which is false.

\( \Box \).
Theorem: There is no set $T$ of sentences such that $T \cup T'$ is finite.

Proof: Recall $T$, $T \cup T'$ is finite.

Suppose, $\forall A: T \cup T' \not\models A$. Then $T \cup T'$ is unsatisfiable.

By compactness, $\exists$ a finite unsatisfiable set $T \cup T' \cup S$, $T \cup S \cup T'$.

Let $B$ be $S \cup T'$, $C$ be $S \cup T'$. $A$ - big "and".

Now $\models T$ iff $A \models C$. $A$ is finite.

$\exists \emptyset$