Completeness Theorem: Γ is a set of sentences, A is a formula

(a) If $\Gamma \vDash A$ then $\Gamma \vdash A$.

(b) If Γ is consistent then Γ is satisfiable.

We prove part (b).

Languages: Γ, A are L-sentences / L-formula.

$$L^{+} = L \cup \{d_1, d_2, d_3, \ldots\} \quad \text{new constant symbols.}$$

Defn A closed term is a term with no variables.

Defn A set Γ is Henkin if for all sentences $\exists x A(x)$,

if $\Gamma \vdash \exists x A(x)$, then $\Gamma \vdash A(t)$ for some closed term t.

t - "witness" for $\exists x A(x)$.

$\exists x A(x)$ means $\neg \forall x \neg A(x)$.

Alternately, Γ is Henkin if whenever $\Gamma \vdash \neg \forall x A(x)$,

then $\Gamma \vdash \neg A(t)$ for some closed term t.

Defn: Γ is strongly Henkin if for all ~~closed~~ sentences $\forall x A(x)$.

$$\Gamma \vdash A(c) \to \forall x A(x)$$

for some constant symbol c.

Lemma If $\forall x A(x)$ is a sentence and c is a "new" constant symbol (not in A, T), then

$$T \cup \{ A(c) \to \forall x A(x) \} \text{ is consistent.}$$

Pf: Suppose this fails, $T \vdash \neg(A(c) \to \forall x A(x))$ by Contradiction Principle.

By TAUT: $T \vdash A(c)$ and $T \vdash \neg \forall x A(x)$

By $T \vdash A(c)$, we have $T \vdash \forall x A(x)$. by Theorem on Constants.

Contradicts consistency of T.

Theorem If T is a consistent set of L-sentences, then there exists a consistent, strongly Henkin set of L^+-sentences, called Δ, such that $\Delta \supseteq T$.

Pf Enumerate all L^+-sentences $\forall x_{i_j} A_j(x_{i_j})$ $j = 1, 2, 3, \ldots$ so that no d_j appears in $\forall x_{i_k} A_k(x_{i_k})$ for $k \leq j$.

Let $T_0 = T$

$$T_{i+1} = T_i \cup \left(A(d_j) \to \forall x_{i_j} A_j(x_{i_j}) \right)$$

By induction, each T_{i+1} is consistent. Set $\Delta = \bigcup T_i$

Lindenbaum Theorem If Δ is consistent and strongly Henkin, there is a consistent, complete and strongly Henkin $\Pi \supseteq \Delta$ of $\mathcal{L}^+$-sentences

Defn Π is complete if for all $\mathcal{L}^+$-sentences A, either $A \in \Pi$ or $\neg A \in \Pi$.

Pf $\Delta_0 = \Delta$ $\quad \Delta_{i+1} = \begin{cases} \Delta_i \cup \{A_i\} & \text{if consistent} \\ \Delta_i \cup \{\neg A_i\} & \text{otherwise.} \end{cases}$

where A_i's enumerate all $\mathcal{L}^+$-sentences $\square$

Properties of Π: A, B - $\mathcal{L}^+$-sentences

$A \in \Pi$ iff $\neg A \notin \Pi$.

$A \to B \in \Pi$ iff $A \notin \Pi$ or $B \in \Pi$

If $\Pi \vdash A$, then $A \in \Pi$.

$\neg \forall x A(x) \in \Pi$ iff for some closed term t, $\neg A(t) \in \Pi$
(some constant d)

Π has a $A(d) \leftrightarrow \forall x A(x)$

Situation 1: Symbol "=" not in language L.

Define $\mathfrak{A}$ by $|\mathfrak{A}|$ = set of closed L^+-terms.

$$c^{\mathfrak{A}} = c$$

$$P^{\mathfrak{A}} = \{\langle t_1, \ldots, t_k\rangle : P(t_1, \ldots, t_k) \in \Pi\}.$$

$$f^{\mathfrak{A}}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$$

Claim $\mathfrak{A} \models \Pi$, hence $\mathfrak{A} \models T$.

$\mathfrak{A}$ is a L^+-structure.

Definition of truth, gives values to more complex formulas.

$$\mathfrak{A} \models \neg P(t_1) \iff \mathfrak{A} \not\models P(t_1) \iff P(t_1) \notin \Pi \iff \neg P(t_1) \in \Pi.$$

$$\mathfrak{A} \models A \to B \underset{\text{def of truth}}{\iff} \mathfrak{A} \not\models A \text{ or } \mathfrak{A} \models B \underset{\text{ind hyp on complexity of formulas}}{\iff} A \notin \Pi \text{ or } B \in \Pi \iff A \to B \in \Pi.$$

$\mathfrak{A} \models \forall x B(x) \iff \mathfrak{A} \models B(t)$ for all L^+-terms t.

$\sigma(t) = t$ all object assignments σ and closed terms t

$\Rightarrow \mathfrak{A} \models B(d)$ for the "Henkin witness" for $\forall x B(x)$

$\Leftrightarrow B(d) \in \Pi$ by induction hypothesis

$\Leftrightarrow \forall x B(x) \in \Pi$ since $B(d) \to \forall x B(x) \in \Pi$.

$\forall x B(x) \in \Pi \Rightarrow$ for all closed terms t, $B(t) \in \Pi$

$\Leftrightarrow \mathfrak{A} \models B(t)$ for all closed terms. by induction hypothesis

$\Leftrightarrow \mathfrak{A} \models B(x)[\sigma]$ for all object assignments σ.

$(\sigma(x) = t$ for a closed term $t.)$

$\Leftrightarrow \mathfrak{A} \models \forall x B(x)$ by defn of truth

So $\mathfrak{A} \models \forall x B(x)$ iff $\forall x B(x) \in \Pi$.

<u>So</u> $\mathfrak{A} \models A$ for all $A \in \Pi$. $\mathfrak{A}$ is a L^+-structure.

$\mathfrak{A} \models \Gamma$ since $\Pi \supseteq \Gamma$.

Not Henkin: Let $\Gamma = \{\exists x P(x)\}$ $\qquad L = \{c, P\}$

$\models \exists x (P(x) \to \forall y P(y))$ - logically valid

$\exists x (P(x) \to \forall y P(y)) \models\!\dashv (\forall x P(x) \to \forall y P(y))$

What if "=" is in the language?

Modify the $|\mathfrak{A}|$

Defn $s \approx t$ if $s = t \in T$.

$\approx$ is an equivalence relation.

$[\![s]\!]$ - $\{t : t \approx s\}$ - the equivalence class that contains s

Define $\mathfrak{A} = \{[\![s]\!] : s \text{ is a closed } L^+ \text{ term}\}$

$$c^{\mathfrak{a}} = [\![c]\!]$$

$$f^{\mathfrak{a}}([\![t_1]\!], \ldots [\![t_k]\!]) = [\![f(t_1, \ldots, t_k)]\!].$$

$$P^{\mathfrak{a}}([\![t_1]\!], \ldots [\![t_k]\!]) = \text{True iff } P(t_1 \ldots, t_k) \in T.$$

Lemma f, P are well-defined.

Theorem $\mathfrak{A} \models B$ iff $B \in T$ for all L^+-sentences B

QED Completeness Theorem.

Compactness Theorem Let Γ be a set of sentences.

1. $\Gamma \models A$ only if $\Gamma' \models A$ for some finite $\Gamma' \subseteq \Gamma$.
2. Γ is unsatisfiable iff some finite subset of Γ is unsatisfiable.

Pf An immediate corollary.

Application: Fix $\mathcal{L}$.

Theorem There is no sentence B s.t. $\forall \mathfrak{A}$,

$\mathfrak{A} \models B$ if and only if $|\mathfrak{A}|$ is finite.

Proof. We'll show no sentence C s.t. $\forall \mathfrak{A}$

$\mathfrak{A} \models C$ if and only if $|\mathfrak{A}|$ is infinite.

Let $T = \{AtLeast_2, AtLeast_3, AtLeast_4, \ldots\}$

$\mathfrak{A} \models T$ iff $\mathfrak{A}$ is infinite.

If C had the above properties, $T \models C$

So some finite $T' \subseteq T$, has $T' \models C$.

But this means $AtLeast_k \models C$ for some finite k.

which is false.

qed.

Theorem there is no set Π of sentences such that $\forall \mathfrak{A}$, $\mathfrak{A} \models \Pi$ iff $|\mathfrak{A}|$ is finite.

Proof: Recall Γ, $\mathfrak{A} \models \Gamma$ iff $\mathfrak{A}$ is infinite.

Suppose, $\forall \mathfrak{A}$; $\mathfrak{A} \models \Pi$ iff $|\mathfrak{A}|$ is finite.

Then: $\Gamma \cup \Pi$ is unsatisfiable.

By Compactness, it has a finite unsatisfiable set

$\Gamma' \cup \Pi'$ $\quad \Gamma' \subseteq \Gamma$, $\Pi' \subseteq \Pi$.

Let B be $\bigwedge \Gamma'$, C be $\bigwedge \Pi'$. $\bigwedge$ - big "and"

Now $\mathfrak{A} \models B$ iff $\mathfrak{A} \not\models C$ iff $\mathfrak{A}$ is finite.

qed