Generalization inference (GEN)

$$\frac{C \to A}{C \to \forall x A}$$

x - not free in C

x is called the eigenvariable.

Deduction Theorem for FO: Assume A is a sentence.

$\Gamma \vdash A \to B$ iff $\Gamma, A \vdash B$

Pf $\Leftarrow$ Easy, use Modus Ponens (A can be general formula)

Somewhat more general theorem.

If $\Gamma, A \vdash B$ with no free variable of A used in the proof as an eigenvariable, then $\Gamma \vdash A \to B$.

Pf: Suppose $B_1, B_2, \ldots, B_k$ is a proof of B from $\Gamma \cup \{A\}$.

Claim: For each i, $\Gamma \vdash A \to B_i$. (This implies the theorem.)

Pf: by induction on i:

Cases (1): B_i is an axiom or a member of Γ. So $\Gamma \vdash B_i$. So $\Gamma \vdash A \to B_i$ by TAUT.

Case (2) B_i is A. Then $A \to A$ is a tautology, so $\Gamma \vdash A \to A$.

Case (3): B_i is inferred by M.P. Ind hyps: $\Gamma \vdash A \to B_j$, $\Gamma \vdash A \to B_j \to B_i$

So by TAUT, $\Gamma \vdash A \to B_i$

Case (4) B_i is inferred by GEN. B_i is $C \to \forall x D$

B_j is $C \to D$, where x is not free in C. (x-eigenvariable)

Ind hyp: $\Gamma \vdash A \to C \to D$

So $\Gamma \vdash A \wedge C \to D$ Taut

So $\Gamma \vdash A \wedge C \to \forall x D$ Gen

So $\Gamma \vdash A \to C \to \forall x D$ Taut.

q.e.d.

Defn Γ is inconsistent iff $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ for some A.

Thm Γ is inconsistent iff $\Gamma \vdash B$ for all formula B.

Thm (Principle of Contradiction) Let A be a sentence.

(a) $\Gamma \vdash A$ iff $\Gamma \cup \{\neg A\}$ is inconsistent.

(b) $\Gamma \vdash \neg A$ iff $\Gamma \cup \{A\}$ is inconsistent.

Thm Let A be a sentence. If Γ is consistent, then at least one of $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is consistent.

Thm (Proof by cases) Let A be a sentence.

If $\Gamma, A \vdash B$ and $\Gamma, \neg A \vdash B$, then $\Gamma \vdash B$.

Theorem on Constants. Let Γ be a set of sentences and let $A(x)$ be a formula and let c be a "new" constant symbol, i.e., c does not appear in Γ or A.

Then (a) $\Gamma \vdash \forall x A(x)$ if and only if $\Gamma \vdash A(c)$

(b) Suppose $\exists x A(x)$ is a sentence. Then

$\Gamma \cup \{\exists x A(x)\}$ is consistent iff $\Gamma \cup \{A(c)\}$ consistent.

(b') $\Gamma, \exists x A(x) \vdash D$ iff $\Gamma, A(c) \vdash D$.

Proof of (a): $\Rightarrow$ Easy. $\forall x A(x) \to A(c)$ is an axiom.

$\Leftarrow$ Let $B_1, \ldots, B_k$ be a proof of $A(c)$ from Γ

Let x_n be a variable not used anywhere in the proof

Let B_i' be B_i with every c replaced by x_n.

Proof by induction: $B_1', \ldots B_k'$ is a valid proof of $A(x_n)$ from Γ! (x_n is not used as an eigenvariable).

So $\Gamma \vdash A(x_n)$

So $\Gamma \vdash \forall x_n A(x_n)$ $\quad$ x_n - not the same as x.

$\Gamma \vdash A(x)$ $\quad$ by UI axiom $\forall x_n A(x_n) \to A(x)$

So $\Gamma \vdash \forall x A(x)$

qed.

Soundness Theorem:

Let A be a formula. Let T be a set of sentences.

(a) If $T \vdash A$ then $T \models A$

(b) If T is satisfiable, then T is consistent.

If T is inconsistent then T is unsatisfiable.

(b) follows from (a) easily, by exactly same proof as was used for PL.

Proof of (a): Let $B_1, B_2, \ldots, B_k$ be a proof of A from T.

Let $\mathfrak{A}$ be a structure, such that $\mathfrak{A} \models T$.

We'll show, for all object assignments σ, $\mathfrak{A} \models B_i[\sigma]$. $i = 1, \ldots k$.

Proof is by induction on i:

Case (1) B_i is an axiom or a $B_i \in T$. So $\mathfrak{A} \models B_i[\sigma]$ (since $\mathfrak{A} \models T$)

Case (2): B_i is inferred by M.P. from B_j and $B_k := B_j \to B_i$ $j, k < i$.

Ind hyp's: $\mathfrak{A} \models B_j[\sigma]$ and $\mathfrak{A} \models B_j \to B_i[\sigma]$. So $\mathfrak{A} \models B_i[\sigma]$.

Case (3): B_i is $C \to \forall x D$ inferred from $B_j := C \to D$, x not free in C.

Ind hyp: $\mathfrak{A} \models C \to D[\sigma]$ for all σ.

therefore $\mathfrak{A} \models \forall x (C \to D)[\sigma]$ for all σ

Since $\forall x (C \to D) \models C \to \forall x D$ by prenex, so $\mathfrak{A} \models (C \to \forall x D)[\sigma]$

qed.

Completeness Theorem (!) Let Γ be a set of formulas and A be a formula.

(a) If $\Gamma \models A$, then $\Gamma \vdash A$.

(b) If Γ is consistent, then Γ is satisfiable.

~~Pro~~ Comment Wlog ~~of~~ we can treat Γ as a set of sentences

Since $\Gamma \vdash A$ iff $\forall \Gamma \vdash A$.

and if $\Gamma \models A$, then $\forall \Gamma \models A$.

Same comments apply (b)

First ingredients of the proof of the Completeness Theorem.

Defn A closed term is a term with no variables.

Ex $0+0$ is closed $0+x_1$ is not.

Defn A set Γ of sentences is Henkin if for every sentence $\exists x A(x)$ s.t. $\Gamma \vdash \exists x A(x)$, there is a closed term t such that $\Gamma \vdash A(t)$.

Defn T is Henkin if for all sentences $\exists x A(x)$

if $T \vdash \exists x A(x)$, then for some closed term, $T \vdash A(t)$.

Defn T is strongly Henkin if for all sentences $\exists x \, A(x)$

there is a constant symbol c such that

$\exists x A(x) \to A(c)$ is in T.

Big lemma: (Henkin / Lindenbaum Theorem)

If T is consistent, then, after introducing new constant symbols $c_1, c_2, c_3, c_4 \ldots$, there is a complete, consistent, strongly Henkin $\Pi \supseteq T$.

Defn Π is complete if for every sentence A, either $A \in \Pi$ or $\neg A \in \Pi$.

Use Π to construct a structure $\mathfrak{A} \models \Pi$.

$|\mathfrak{A}|$ = set of constants (sort-of!)