Prenex formulas:

Recall a formula is prenex if it has the form

$$Qx_{i_1}\, Qx_{i_2} \cdots Qx_{i_k}\, B \qquad B \text{ - quantifier-free}$$

Bunch of equivalences - if x is not free in C:

$\forall x (A \vee C) \vDash\dashv \forall x A \vee C$

$\exists x (A \vee C) \vDash\dashv \exists x A \vee C$

$\forall x (A \wedge C) \vDash\dashv \forall x (A \wedge C)$

$\exists x (A \wedge C) \vDash\dashv \exists x A \wedge C$

$\star'$ $\forall x A \to C \vDash\dashv \exists x (A \to C)$

$\star$ $\exists x A \to C \vDash\dashv \forall x (A \to C)$

$C \to \forall x A \vDash\dashv \forall x (C \to A)$

$C \to \exists x A \vDash\dashv \exists x (C \to A)$

none for $\leftrightarrow$ use $(A \to B) \wedge (B \wedge A)$

These equivalences plus alphabetic variants allow any formula to be put in prenex form.

Example $\exists x\, Q(x,y) \to \forall x \exists y\, P(x,y)$ ← This is A

$A \vDash\dashv \forall x (Q(x,y) \to \forall x \exists y\, P(x,y))$ by above $\star$

$\vDash\dashv \forall x (Q(x,y) \to \forall x' \exists y\, P(x',y))$ alphabetic variant

$\vDash\dashv \forall x \forall x' (Q(x,y) \to \exists y\, P(x',y))$ by above $\star'$

$\vDash\dashv \forall x \forall x' \exists y' (Q(x,y) \to P(x',y'))$ alphabetic variant and $\star$.

First-order proof system FO

- Hilbert-style system
- Axioms & 2 rules of inference
- Step-by-step ~~reasoning~~ reasoning
- Notation: $\vdash A$ or $\Gamma \vdash A$ means A has an FO-proof (from Γ) \vdash
- Soundness & Completeness (Γ - set of sentences)
- Compactness Theorem

- Use only connectives $\neg, \rightarrow, \forall$

$A \vee B$, $A \wedge B$ mean $\neg A \rightarrow B$, $\neg(A \rightarrow \neg B)$ } abbreviations
$\exists x A$ means $\neg \forall x \neg A$

Axioms:

① PL-axioms Let A, B, C be formulas

$$A \to (B \to A) \qquad (A \to (B \to C)) \to (A \to B) \to (A \to C)$$

$$\neg A \to (A \to B) \qquad (\neg A \to A) \to A.$$

② Equality axioms - x, y, z, x_i, y_i - arbitrary variables

$x = x$ Reflexivity

$x = y \to y = x$ Symmetry

$x = y \to y = z \to x = z$ Transitivity.

See later slide for missing equality axioms

③ $\forall x\, A(x) \to A(t)$ - using informal "relaxed" notation for substitution

i.e. $\forall x A \to A(t/x)$ provided t is substitutable for x in A

Rules of inference:

(1) Modus Ponens
$$\frac{A \qquad A \to B}{B}$$

(2) Generalization
$$\frac{C \to A}{C \to \forall x A}$$
provided x does not appear free in C.

<u>Definition</u> Let T be a set of formulas.

An <u>FO-proof</u> of a formula A from T, is a sequence of formulas ~~...~~

$$C_1, C_2, \ldots, C_k$$

such that: C_k is A and each C_i satisfies:

(1) C_i is an axiom or $C_i \in T$, or

(2) C_i inferred by Modus Ponens from C_j, C_k, $j, k < i$

(3) C_i " " Generalization from C_j, for some $j < i$.

We write $T \vdash A$ for "A has an FO-proof from T".

<u>Definition</u> A FO-proof of A is an FO-proof of A from $\emptyset$.

We write $\vdash A$.

<u>Generalization</u> $\dfrac{C \to A}{C \to \forall x A}$ Stronger form of $\dfrac{A}{\forall x A}$

Example: $P(x) \vdash \forall x P(x)$ $\qquad T = \{P(x)\}$

Proof:

$P(x) \vdash P(x)$ — Hypothesis (in T)

$P(x) \vdash P(x) \to y=y \to P(x)$ — PL1

$P(x) \vdash y=y \to P(x)$ — M.P. $\qquad (1)$

$P(x) \vdash y=y$ — Axiom (Reflexivity)

$P(x) \vdash y=y \to \forall x P(x)$ — Generalization from (1)

$P(x) \vdash \forall x P(x)$ — M. P.

Theorem $A \vdash \forall x A$ ~~and~~ and if $T \vdash A$ then $T \vdash \forall x A$

Proof by above construction.

Also note PL axioms and Modus Ponens allow general propositional reasoning.

Definition: We call $\forall x A$ a generalization of A

Let x_{i_1}, x_{i_k} be the variables that occur free in A.

then $\underbrace{\forall x_{i_1} \cdots \forall_{i_k} A}$ is called the universal closure of A.

this is a sentence. - denoted $\forall(A)$

Corollary If B is a generalization ~~of~~ or a universal closure of A and $T \vdash A$, then $T \vdash B$.

Definition: $\forall T$ is the set $\{\forall(B) : B \in T\}$.

Corollary $T \vdash A$ if and only if $\forall T \vdash A$.

~~Pf~~: So w/og we can T to be a set of sentences.

Proof: $\Rightarrow$ Suppose $T \vdash A$.

Consider $B \in T$. Claim $\forall T \vdash B$

Pf of claim $\vdash \forall x D \to D$ is an axiom for any formula D

So by MP, $\forall T \vdash B$, for all $B \in T$.

$\Leftarrow$ For each $B \in T$, $T \vdash \forall(B)$ - by 1ST corollary.

Note $P(x) \vdash P(y)$

But $P(x) \not\models P(y)$

Example Show $P(x) \vdash Q(z) \to P(x)$

PL1	$P(x) \to Q(z) \to P(x)$	Axiom
	$P(x)$	Hypothesis
	$Q(z) \to P(x)$	Modus Ponens

Theorem: (a) If A is a tautology, then $\vdash A$

(b) If T tautologically implies A, then $T \vdash A$.

Corollary Modus Tollens and Hypothetical Syllogism are derived (admissable) ~~rul~~ rules of inference ~~for~~ for FO.

$$\frac{A \to B \qquad B \to C}{A \to C}$$

By Theorem

$\vdash (A \to B) \to (B \to C) \to (A \to C)$

Proof (a) Suppose A is a tautology. I.e. $A = B(C_1 \dots C_k / p_1 \dots p_k)$

where $B = B(p_1 \dots p_k)$ is a propositional tautology.

Therefore, by Completeness for PL, there is a PL-proof

$D_1, \dots, D_\ell$ of B.

Then $D_1(\vec{C}/p), D_2(\vec{C}/\vec{p}) \dots, D_\ell(\vec{C}/\vec{p})$ is a valid FO-proof of A.

(b) Immediate from (a)

If T tautologically implies A, then

($\star$) $B_1 \to B_2 \to \dots \to B_k \to A$ is a tautology for some $B_1 \dots B_k \in T$

By (a), it has a FO-proof, and then use HYP and MP

k-times.

$\boxed{\checkmark}$

Recall: T tautologically implies A <u>means</u> (by definition)

$\exists$ finitely many $B_1, \dots B_k \in T$ s.t. ($\star$) is a tautology.

Example

$$\vdash \; x = y \to g(f(x), z) = g(f(y), z).$$

Proof.

$\vdash \; x = \cancel{y} \to f(x) = f(y)$ Axiom

$\vdash \; u = v \to z = z \to g(u, z) = g(v, z)$ Axiom.

$\vdash \; \forall u \forall v \forall z\, (u = v \to z = z \to g(u, z) = g(v, z))$

$\boxed{\text{Using } \dfrac{D}{\forall x D} \text{ 3 times}}$

$\vdash \; f(x) = f(y) \to z = z \to g(f(x), z) = g(f(y), z))$

Used $\forall x A \to A(t/x)$, t substitutable

(1) t is $f(x)$ $\cancel{\ldots}$

(2) t is $f(y)$

(3) t is z.

$\vdash \; z = z$ Axiom

$\vdash \; x = y \to g(f(x), z) = g(f(y), z).$

Missing equality axioms:

$$x_1 = y_1 \to x_2 = y_2 \to \ldots \; x_k = y_k \to f(x_1 \ldots x_k) = f(y_1 \ldots y_k)$$

$$x_1 = y_1 \to x_2 = y_2 \to \ldots \; x_k = y_k \to P(x_1 \ldots x_k) \to P(y_1, \ldots, y_k)$$

Similarly

$$\vdash f(x) = f(x)$$

Pf:

$\vdash y = y$ Axiom

$\vdash \forall y (y = y)$ Generalization Thm

$\vdash \forall y (y = y) \to f(x) = f(x)$ Axiom

$\vdash f(x) = f(x)$ MP

Deduction Theorem $\Gamma, A \vdash B$ iff $\Gamma \vdash A \to B$

where A is a sentence