Recall substitution: $A(t/x)$

"$t$ is substitutable for $x$ in $A$" - when it makes sense to form $A(t/x)$.

**Application 1:** Alphabetic variant of $A$.

Renaming bound variables:

$\exists x_2 \left( x_2 + x_2 = x_1 \right) =: A$

$A \equiv B$

$\exists x_3 \left( x_3 + x_3 = x_1 \right) =: B$

**Application 2**

**Theorem:** If $t$ is substitutable for $x$ in $A$, then

$\vdash \forall x A \rightarrow A(t/x)$.

**Example:** $A$ as above $A(z+w/x)$ is $\exists x_2 \left( x_2 + x_2 = z + w \right)$

Theorem states: $\vdash \forall x, \exists x_2 / x_2 + x_2 = x_1 \rightarrow \exists x_2 / x_2 + x_2 = z + w$

What if $t$ is not substitutable for $x$ in $A$? Can it go wrong?

Yes! **Example:** Let $A$ be $\exists y (y + 1 = x)$. Let $t$ be $y$.

Then $\forall x \exists y (y + 1 = x) \rightarrow \exists y (y + 1 = x)$ is not logically valid.
Proof: Wlog \( x \) does not appear int. 
Otherwise use an alphabetic variant \( \forall x B(x) \Rightarrow \forall x A(x) \)

Then
\[ \vdash x = t \rightarrow (A \rightarrow A(t/x)) \]
\[ \vdash x = t \rightarrow A \rightarrow A(t/x) \]
\[ \vdash \forall x A \rightarrow A \]
\[ \vdash x = t \rightarrow \forall x A \rightarrow A(t/x) \]

By last week's Substitution Theorem by Tautological Implication:

Special case of the theorem being proved:

- Easy to prove directly: Tautology.

Generalization

If \( \vdash B \) then \( \vdash \forall x B \)

Need

\[ \forall x (A \rightarrow C) \vdash \exists x A \rightarrow C \]
if \( x \) is not free in \( C \).

Pf: Take \( \sigma(x) = \sigma(t) \)

I.e. let \( T \) be the \( x \)-variant of \( \sigma \) with \( T(x) = \sigma(t) \)

Then \( \forall \sigma \vdash \exists x (x = t) [\sigma] \)
Theorem: Suppose $\mathcal{E} \vdash \alpha$. Then $\mathcal{E} \vdash \forall x \alpha$ (Justifies the Generalization)

**Proof:** Suppose $\mathcal{E} \vdash \alpha$.

Given $\mathcal{M}, \sigma$ we need to show $\mathcal{M}, \sigma \models \forall x \alpha [\sigma]$, i.e., that $\mathcal{M}, \sigma \models \alpha [\tau]$ for all $x$-variants $\tau$ of $\sigma$.

That's immediate from $\mathcal{E} \vdash \alpha$, since thus $\mathcal{M}, \sigma \models \alpha [\tau]$.

Therefore.

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**Theorem:** Suppose $x$ in not free in $C$. Then:

(a) $\forall x (C \rightarrow A) \vdash \forall x (C \rightarrow A)$.

(b) $\forall x (A \rightarrow C) \vdash \exists x A \rightarrow C$.

**Proof of (a):** Let $\mathcal{M}, \sigma$ be given. Want to show $\mathcal{M}, \sigma \models \forall x (C \rightarrow A) [\sigma]$ if and only if $\mathcal{M}, \sigma \models C \rightarrow \forall x A [\sigma]$.

**Case (1):** $\mathcal{M} \not\models C [\sigma]$. So $\mathcal{M}, \sigma \not\models C \rightarrow \forall x A [\sigma]$.

Also, $\mathcal{M}, \sigma \not\models C [\tau]$ for all $x$-variants $\tau$ of $C$. (Because $x$ not free in $C$.)

So $\mathcal{M}, \sigma \models C \rightarrow A [\tau]$ " " " " " " " " " " " " " " free in $C$.

So $\mathcal{M}, \sigma \models \forall x (C \rightarrow A) [\sigma]$ by definition of truth.

**Case (2):** $\mathcal{M}, \sigma \models C [\sigma]$. 


Case 2: \( \alpha \vdash C[\alpha] \).

\[ \alpha \vdash C \rightarrow \forall x \, A[\alpha], \text{iff} \quad \alpha \vdash \forall x \, A[\alpha]. \]

\[ \alpha \vdash \forall x \,(C \rightarrow A)[\alpha], \text{iff} \quad \text{for all } x\text{-variant } \tau \text{ of } \alpha, \quad \alpha \vdash C \rightarrow A[\tau], \]

\[ \text{iff} \quad \alpha \vdash A[\tau], \quad \text{since } \alpha \vdash C[\tau] \text{ for these } \tau \text{'s.} \]

\[ \text{iff} \quad \alpha \vdash \forall x \, A[\alpha]. \]

\[ \text{iff} \quad \alpha \vdash \forall x \, A[\alpha]. \quad \text{good part (a) } \square \]

Proof of (6):

\[ \forall x \,(A \rightarrow C) \vdash \forall x \, (\neg C \rightarrow \neg A), \]

\[ \vdash \neg C \rightarrow \forall x \, \neg A, \quad \text{by part (a), Tauf} \]

\[ \vdash \neg \forall x \, \neg A \rightarrow C \]

\[ \vdash \exists x \, A \rightarrow C. \quad \square \]
Prenex operators  C - does not contain a free occurrence of

\[ \forall x (A \rightarrow C) \equiv \exists x A \rightarrow C \]
\[ \forall x (C \rightarrow A) \equiv C \rightarrow \forall x A \]
\[ \exists x (A \rightarrow C) \equiv \forall x A \rightarrow C \]
\[ \exists x (C \rightarrow A) \equiv C \rightarrow \exists x A \]

\[ \exists x (A \land C) \equiv \exists x A \land C \]
\[ \forall x (A \land C) \equiv \forall x A \land C . \quad \text{-- easier.} \]

\[ \exists x (A \lor C) \equiv \exists x A \lor C \]
\[ \forall x (A \lor C) \equiv \forall x A \lor C . \]
\[ \neg \forall x A \equiv \exists x \neg A \]
\[ \neg \exists x A \equiv \forall x \neg A \]

Defn A formula is in prenex form if it has the

form  \[ Q x_{i_1}, Q x_{i_2} \ldots Q x_{i_k} B \] , where B has

no quantifiers.
Proofs in first-order logic

Axioms: PL1 - PL9

Equality:

- Reflexivity: \( x = x \)
- Symmetry: \( x = y \rightarrow y = x \)
- Transitivity: \( x = y \rightarrow y = z \rightarrow x = z \)

Equality of Functions:

- \( x_1 = y_1 \rightarrow x_2 = y_2 \rightarrow \ldots \rightarrow x_n = y_n \rightarrow f(x_1, x_2, \ldots, x_n) = f(y_1, y_2, \ldots, y_n) \)

Universal Elimination:

\( \forall x A \rightarrow A(t/x) \) provided \( t \) is substitutable for \( x \) in \( A \).

Rules of inference:

- Modus Ponens: \( \frac{A \quad A \rightarrow B}{B} \)
- Generalization:
  \( \frac{C \rightarrow A}{C \rightarrow \forall x A} \) provided \( x \) is not free in \( C \).

Recall:

- \( \frac{A}{\forall x A} \)
- \( \frac{C \rightarrow A}{\forall x (C \rightarrow A)} \)
- \( \forall x (C \rightarrow A) \rightarrow C \rightarrow \forall x A \)
- \( C \rightarrow \forall x A \)