

Recall substitution:  $A(t/x)$

" $t$  is substitutable for  $x$  in  $A$ " - when it makes sense to form  $A(t/x)$ .

Application: Alphabetic variant of  $A$ .

Renaming bound variables

$$\exists x_2 (x_2 + x_2 = x_1) =: A$$

$$A \neq B.$$

$$\exists x_3 (x_3 + x_3 = x_1) =: B$$

Application # 2

Theorem If  $t$  is substitutable for  $x$  in  $A$ , then  
 $\models \forall x A \rightarrow A(t/x)$ .

Example  $A$  as above  $A(z+u/x)$  is  $\exists x_2 (x_2 + x_2 = z+u)$

Theorem states:  $\models \forall x, \exists x_2 (x_2 + x_2 = x_1) \rightarrow \exists x_2 (x_2 + x_2 = z+u)$

What if  $t$  is not substitutable for  $x$  in  $A$ ? Can it go wrong?

Yes! Example Let  $A$  be  $\exists y (y+1=x)$ . Let  $t$  be  $y$ .

Then  $\forall x \exists y (y+1=x) \rightarrow \exists y (y+1=y)$  is not logically valid.

Proof: Wlog  $x$  does not appear int.

otherwise use an alphabetic variant  $\forall z B(z)$  or  $\forall x A(x)$

Then  $\vDash x=t \rightarrow (A \leftrightarrow A(t/x))$

by last week's Substitution Thm

$\vDash x=t \rightarrow A \rightarrow A(t/x)$

by Tautological Implication

$\vDash \forall x A \rightarrow A$

Special case of the theorem being proved  
- easy to prove directly

$\vDash x=t \rightarrow \forall x A \rightarrow A(t/x)$

Tautology.

$\vDash \forall x (x=t \rightarrow \forall x A \rightarrow A(t/x))$

Generalization  
If  $\vDash B$  then  $\vDash \forall x B$

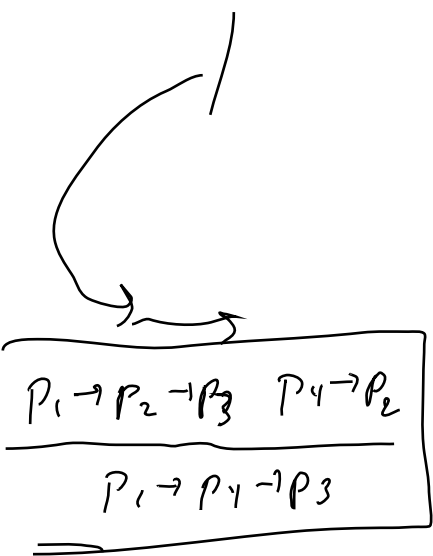
$\Rightarrow \vDash \exists x (x=t) \rightarrow (\forall x A \rightarrow A(t/x))$

Need  
 $\forall x (A \rightarrow C) \vDash \exists x A \rightarrow C$   
if  $x$  is not free in  $C$ .

$\Rightarrow \vDash \exists x (x=t)$

$\vDash \forall x A \rightarrow A(t/x)$   
~~Taut~~

Pf Take  $\sigma(x) = \sigma(t)$   
I.e. let  $\tau$  be the  $x$ -variant of  $\sigma$   
with  $\tau(x) = \sigma(t)$   
Then  $\alpha \vDash \exists x (x=t) [\sigma]$







## Prenex operators

$C$  - does not contain a free occurrence of  $x$

$$\forall x (A \rightarrow C) \equiv \exists x A \rightarrow C$$

$$\forall x (C \rightarrow A) \equiv C \rightarrow \forall x A$$

$$\exists x (A \rightarrow C) \equiv \forall x A \rightarrow C$$

$$\exists x (C \rightarrow A) \equiv C \rightarrow \exists x A$$

$$\exists x (A \wedge C) \equiv \exists x A \wedge C$$

$$\forall x (A \wedge C) \equiv \forall x A \wedge C.$$

- easier.

$$\exists x (A \vee C) \equiv \exists x A \vee C$$

$$\forall x (A \vee C) \equiv \forall x A \vee C.$$

$$\neg \forall x A \equiv \exists x \neg A$$

$$\neg \exists x A \equiv \forall x \neg A.$$

Defn A formula is in prenex form if it has the

form  $Qx_1, Qx_2, \dots, Qx_k B$ , where  $B$  has no quantifiers.

Proofs in first-order logic

$\neg, \rightarrow, \forall$  - logical connectives

Axioms

PL1 - PL4

$$A \rightarrow (B \rightarrow A)$$

Equality

$$x = x$$

reflexivity

$$x = y \rightarrow y = x$$

symmetry

$$x = y \rightarrow y = z \rightarrow x = z$$

transitivity

$$x_1 = y_1 \rightarrow x_2 = y_2 \rightarrow \dots \rightarrow x_k = y_k \rightarrow f(x_1, \dots, x_k) = f(y_1, \dots, y_k)$$

$$x_1 = y_1 \rightarrow \dots \rightarrow x_k = y_k \rightarrow P(x_1, \dots, x_k) \rightarrow P(y_1, \dots, y_k).$$

( $\leftarrow$ )

$$\forall x A \rightarrow A(t/x)$$

provided  $t$  is substitutable for  $x$  in  $A$ .

Rules of inference:

Modus Ponens 
$$\frac{A \quad A \rightarrow B}{B}$$

Generalization:

$$\boxed{\frac{C \rightarrow A}{C \rightarrow \forall x A}}$$

provided  $x$  is not free in  $C$ .

Recall 
$$\frac{A}{\forall x A}$$

$$\frac{\frac{C \rightarrow A}{\forall x (C \rightarrow A)} \quad \forall x (C \rightarrow A) \rightarrow C \rightarrow \forall x A}{C \rightarrow \forall x A}$$