Recall substitution: $A(t/x)$

"t is substitutable for x in A" - when it makes sense to form $A(t/x)$.

__Application__: Alphabetic variant of A.

Renaming __bound variables__ $\quad \exists x_2 (x_2 + x_2 = x_1) =: A$

$A \models\!\dashv B$. $\quad \exists x_3 (x_3 + x_3 = x_1) =: B$

__Application #2__

__Theorem__ If t is substitutable for x in A, then

$$\models \forall x A \to A(t/x).$$

__Example__ A as above $A(z+u/x_1)$ is $\exists x_2 (x_2 + x_2 = z+u)$

Theorem states: $\models \forall x_1 \exists x_2 (x_2 + x_2 = x_1) \to \exists x_2 (x_2 + x_2 = z + u)$

What if t is __not__ substitutable for x in A? Can it go wrong?

__Yes!__ __Example__ Let A be $\exists y (y+1 = x)$. Let t be y.

Then $\forall x \exists y (y+1 = x) \to \exists y (y+1 = y)$ is not logically valid.

Proof: $\boxed{\text{Wlog } x \text{ does not appear in } t.}$

Otherwise use an alphabetic variant $\forall z B(z)$ for $\forall x A(x)$

Then

$\models x = t \to (A \leftrightarrow A(t/x))$ by last week's Substitution Thm

$\models x = t \to A \to A(t/x)$ by Tautological Implication

$\models \forall x A \to A$ Special case of the theorem being proved – easy to prove directly

$\models x = t \to \forall x A \to A(t/x)$ Tautology.

$$\boxed{\dfrac{p_1 \to p_2 \to p_3 \quad p_4 \to p_2}{p_1 \to p_4 \to p_3}}$$

$\models \forall x\,(x = t \to \forall x A \to A(t/x))$

$\boxed{\text{Generalization: If } \models B \text{ then } \models \forall x B}$

$\Rightarrow \models \exists x (x = t) \to \underbrace{(\forall x A \to A(t/x))}$

$\boxed{\text{Need } \forall x (A \to C) \models \exists x A \to C \text{ if } x \text{ is not free in } C.}$

$\Rightarrow \models \exists x (x = t)$

Pf Take $\sigma(x) = \sigma(t)$
I.e. let τ be the x-variant of σ with $\tau(x) = \sigma(t)$
Then $\mathfrak{A} \models \exists x (x = t)[\sigma]$

$\models \forall x A \to A(t/x)$ Taut

<u>Theorem</u> Suppose $\models A$. Then $\models \forall x A$ (Justifies the generalization)

<u>Pf</u> Suppose $\models A$.

Given $\mathfrak{A}, \sigma$ we need to show $\mathfrak{A} \models \forall x A[\sigma]$.

i.e., that $\mathfrak{A} \models A[\tau]$ for all x-variants τ of σ.

That's immediate from $\models A$, since thus $\mathfrak{A} \models A[\tau]$

<u>Qed.</u>

<u>Theorem</u> Suppose x is not free in C. Then:

(a) $\forall x (C \to A) \models\dashv C \to \forall x A$.

(b) $\forall x (A \to C) \models\dashv \exists x A \to C$.

<u>Pf of (a)</u>: Let $\mathfrak{A}, \sigma$ be given. Want to show $\mathfrak{A} \models \forall x (C \to A)[\sigma]$ if and only if $\mathfrak{A} \models C \to \forall x A[\sigma]$.

<u>Case (1)</u> $\mathfrak{A} \not\models C[\sigma]$. So $\mathfrak{A} \models C \to \forall x A[\sigma]$.

Also, $\mathfrak{A} \not\models C[\tau]$ for all x-variants of C. (Because x not free in C.)

So $\mathfrak{A} \models C \to A[\tau]$ " " " " "

So $\mathfrak{A} \models \forall x (C \to A)[\sigma]$ by definition of truth

<u>Case (2)</u> $\mathfrak{A} \models C[\sigma]$.

Case (2) $\mathfrak{A} \nvDash C[\sigma]$.

$\mathfrak{A} \vDash C \to \forall x A[\sigma]$ iff $\mathfrak{A} \vDash \forall x A[\sigma]$.

$\mathfrak{A} \vDash \forall x (C \to A)[\sigma]$ ~~iff~~

iff for all x-variants τ of σ, $\mathfrak{A} \vDash C \to A[\tau]$

iff " " " " ", $\mathfrak{A} \vDash A[\tau]$

since $\mathfrak{A} \nvDash C[\tau]$ for these τ's.

(since x is not free in C)

iff $\mathfrak{A} \vDash \forall x A[\sigma]$.

qed part (a) $\square$

Proof of (b)

$\forall x (A \to C) \vDash\dashv \forall x (\neg C \to \neg A)$

$\vDash\dashv \neg C \to \forall x \neg A$ by part (a)

$\vDash\dashv \neg \forall x \neg A \to C$ Taut.

$\vDash\dashv \exists x A \to C$.

$\square$

Prenex operations $\qquad$ C - does not contain a free occurrence of x

$$\forall x (A \to C) \vDash\dashv \exists x A \to C$$

$$\forall x (C \to A) \vDash\dashv C \to \forall x A$$

$$\exists x (A \to C) \vDash\dashv \forall x A \to C$$

$$\exists x (C \to A) \vDash\dashv C \to \exists x A$$

$$\exists x (A \wedge C) \vDash\dashv \exists x A \wedge C$$

$$\forall x (A \wedge C) \vDash\dashv \forall x A \wedge C .$$ - easier.

$$\exists x (A \vee C) \vDash\dashv \exists x A \vee C$$

$$\forall x (A \vee C) \vDash\dashv \forall x A \vee C .$$

$$\neg \forall x A \vDash\dashv \exists x \neg A$$

$$\neg \exists x A \vDash\dashv \forall x \neg A .$$

Defn A formula is in prenex form if it has the form $Qx_{i_1} Qx_{i_2} \cdots Qx_{i_k} B$, where B has no quantifiers.

Proofs in first-order logic

$\neg, \to, \forall$ – logical connectives

Axioms

PL1 – PL4 $\qquad A \to (B \to A)$

Equality

$x = x$ reflexivity

$x = y \to y = x$ symmetry

$x = y \to y = z \to x = z$ transitivity

$x_1 = y_1 \to x_2 = y_2 \to \dots x_k = y_k \to f(x_1 \dots x_k) = f(y_1 \dots y_k)$

$x_1 = y_1 \to \dots \to x_k = y_k \to P(x_1 \dots x_k) \to P(y_1 \dots y_k)$.

$(\leftarrow)$

$\forall x A \to A(t/x)$ provided t is substitutable for x in A.

Rules of inference:

Modus Ponens $\dfrac{A \quad A \to B}{B}$

Generalization:

$$\boxed{\dfrac{C \to A}{C \to \forall x A}}$$

provided x is not free in C.

Recall $\dfrac{A}{\forall x A}$

$$\dfrac{\dfrac{C \to A}{\forall x (C \to A)} \qquad \forall x (C \to A) \to C \to \forall x A}{C \to \forall x A}$$