Logical Validity & Logical Implication

Defn $\mathfrak{A}$ is a <u>model</u> of a sentence A, or $\mathfrak{A}$ <u>satisfies</u> A, if $\mathfrak{A} \models A$.

$\mathfrak{A}$ is a <u>model</u> of a set Γ of sentences, or $\mathfrak{A}$ <u>satisfies</u> Γ, if $\mathfrak{A} \models B$ for all $B \in \Gamma$.

Defn $\models A$, A is <u>logically valid</u>, if for all structures $\mathfrak{A}$, $\mathfrak{A} \models A$.

$\mathfrak{A}$ frakfur

$\Gamma \models A$, Γ <u>logically implies</u> A, if for all structures, if $\mathfrak{A} \models \Gamma$ then $\mathfrak{A} \models A$.

Here Γ, A are still (sets of) sentences (not formulas)

Example

$$\models \forall x \forall y\, E(x,y) \rightarrow \forall y \forall x\, E(x,y)$$

$$\forall x\,(x \cdot x = x) \models \forall x\,((x \cdot x) \cdot (x \cdot x) = x)$$

Now A is a formula & T is a set of formulas.

Defn: The pair $(\mathfrak{A}, \sigma)$ satisfies A if $\mathfrak{A} \models A[\sigma]$.

The pair $(\mathfrak{A}, \sigma)$ satisfies T if $\mathfrak{A} \models B[\sigma]$ for all $B \in T$.

We can also say "A is satisfiable" or "T is satisfiable".

Defn: A is logically valid, $\models A$, if for ~~some~~ all pairs $(\mathfrak{A}, \sigma)$,

$$\mathfrak{A} \models A[\sigma].$$

T logically implies A, written $T \models A$, if for all pairs $(\mathfrak{A}, \sigma)$, if $\mathfrak{A} \models B[\sigma]$ for all $B \in T$, then $\mathfrak{A} \models A[\sigma]$.

Examples

$\models \forall x\, (P(x) \to P(x))$.

$\models P(x) \to P(x)$ $\Leftarrow$

$P(x) \models P(x)$

$P(x),\ P(x) \to Q(x) \models Q(x)$ $\leftarrow$ Example of tautological implication.

$P(x) \not\models P(y)$

Example $|\mathfrak{A}| = \{0, 1\}$ $P^{\mathfrak{A}} = \{0\}$.

$\sigma(x) = 0$ $\sigma(y) = 1$

$P(x) \not\models \forall x\, P(x)$

Definition $A \models\!\!\dashv B$, A is logically equivalent to B, if $A \models B$ and $B \models A$.

Example of a logical implication

$$\forall x\, P(x) \models P(y)$$

$$P(x) \models \exists y\, P(y)$$

$$\exists y \forall x\, P(x,y) \models \forall x \exists y\, P(x,y)$$

Universe is always non-empty

Given any $\mathfrak{A}$, any σ.

Suppose $\mathfrak{A} \models \forall x\, P(x)[\sigma]$

i.e. for any x-variant τ of σ, $\mathfrak{A} \models P(x)[\tau]$

$\sigma(y) \in |\mathfrak{A}|$

Let τ be in particular, the x-variant of σ with ~~$\tau(y) = \sigma(x)$~~. $\tau(x) = \sigma(y)$

y is a variable x_j

Since $\mathfrak{A} \models P(x)[\tau]$, $\tau(x) \in P^{\mathfrak{A}}$

Thus $\sigma(y) \in P^{\mathfrak{A}}$ since $\tau(x) = \sigma(y)$.

So $\mathfrak{A} \models P(y)[\sigma]$.

$$\forall x \exists y\, P(x,y) \not\models \exists y \forall x\, P(x,y)$$

$|\mathfrak{A}| = \{0, 1\}$

$P^{\mathfrak{A}} = \{\langle 0,0\rangle, \langle 1,1\rangle\}$.

$\models x = x$

$\models \exists x(x = x)$

Theorem: [Semantic Principle of Contradiction]

(a) $\models A$ if and only if $\{\neg A\}$ is unsatisfiable.

(b) $\Gamma \models A$ if and only if $\Gamma \cup \{\neg A\}$ is unsatisfiable.

Pf (a) is just (b) with $\Gamma = \emptyset$.

(b), $\Gamma \models A$ iff $\forall \mathfrak{A}, \sigma$, if $\mathfrak{A} \models B[\sigma]$ for all $B \in \Gamma$, then $\mathfrak{A} \models A[\sigma]$

$\Gamma \cup \{\neg A\}$ is unsatisfiable iff $\forall (\mathfrak{A}, \sigma)$, it is not the case that $\mathfrak{A} \models B[\sigma]$ for all B in Γ and $\mathfrak{A} \models \neg A[\sigma]$.

These are equivalent. $\boxtimes$

Theorem [Semantic Deduction Theorem]

(a) $\models A \to B$ if and only if $A \models B$

(b) $\Gamma \models A \to B$ if and only if $\Gamma, A \models B$.

Pf (a) is a special case of (b), $\Gamma = \emptyset$.

(b) has an easy proof similar to the above proofs $\boxtimes$

Tautologies (in first-order logic)

Let A be a first-order formula.

<u>Def'n</u> A is a <u>tautology</u> if there is a <u>propositional tautology</u> B with variables $p_1, \dots p_k$ and there are first-order formulas $C_1, \dots, C_k$ so that $B(C_1, C_2, \dots C_k / p_1, \dots, p_k)$ is the same formula as A.

$p_1 \to (p_2 \to p_1)$

<u>Example</u> $\underbrace{P(x)}_{p_1} \to \underbrace{Q(x)}_{p_2} \to \underbrace{P(x)}_{p_1}$ is a tautology.

Take B to be $p_1 \to p_2 \to p_1$

C_1 to be $P(x)$

C_2 to be $Q(x)$

<u>Theorem</u> If A is a tautology (A is a first-order formula), then $\models A$, A is logically valid.

<u>Proof:</u> Fix $\mathfrak{A}, \sigma$. Need to show $\mathfrak{A} \models A[\sigma]$.

Let φ be the <u>truth</u> assignment s.t. $\varphi(p_i) = \begin{cases} T & \text{if } \mathfrak{A} \models C_i[\sigma] \\ F & \text{if } \mathfrak{A} \not\models C_i[\sigma] \end{cases}$

<u>Claim</u> $\varphi(B) = T$ iff $\mathfrak{A} \models A[\sigma]$

$\varphi(D) = T$ iff $\mathfrak{A} \models D(C_1 \dots C_k / p_1 \dots p_k)[\sigma]$ for all subformulas D of B.

Examples $P(x) \wedge Q(x) \to Q(x) \wedge P(x)$ is a tautology

$p_1 \wedge p_2 \to p_2 \wedge p_1$

$P(x) \wedge Q(x) \to Q(x) \vee P(x)$ is a tautology

$p_1 \wedge p_2 \to p_2 \vee p_1$

$\underbrace{\forall x (P(x) \to P(x))}_{p_1}$ is not a tautology

$x = y \to y = x$ is not a tautology.

$p_1 \to p_2$

$\exists x P(x) \to \exists y P(y)$ is not a tautology

$p_1 \to p_2$

Defn T tautologically implies A if there are $B_1, \dots, B_\ell \in T$ s.t. $B_1 \to B_2 \to \cdots \to B_\ell \to A$ is a tautology.

equivalently $B_1 \wedge B_2 \wedge \dots \wedge B_\ell \to A$ is a tautology.

Example $P(x) \wedge Q(x)$ tautologically implies $Q(x) \vee P(x)$.

$\exists x P(x) \to \exists x P(x)$ is a tautology.

$p_1 \to p_1$

$\exists x P(x)$ is called a alphabetic variant of $\exists y P(y)$

<u>Theorem</u>: Let A be a formula.

Let B be a subformula of A.

Let $C \vDash\dashv B$ (C is logically equivalent to B).

Let A^* be obtained from A by replacing B with C.

Then $A \vDash\dashv A^*$ (i.e. A is logically equivalent to A^*).

<u>Defn</u> A is ~~tautologically~~ logically equivalent to B, $A \vDash\dashv B$, if $A \vDash B$ and $B \vDash A$. (A, B are first-order formulas)

<u>Example</u>: $\forall x\, A \vDash\dashv \forall x\, \neg\neg A$ since $A \vDash\dashv \neg\neg A$

the latter holds since A is tautologically equivalent to A.

$$\forall x\, \exists y\, (P(x) \to Q(y)) \vDash\dashv \forall x\, \exists y\, (\neg Q(y) \to \neg P(x))$$