Semantics of first-order logic (Already did syntax.)

Propositional logic: Truth Assignments

First-order logic: Structures or Interpretations

Example: Language of Groups. (additive)

Symbols: 0 - constant symbol

$+$ - binary function (addition ~~multiplication~~)

$-(\cdot)$ - unary function (inverse)

A group is a structure:

A distinguished element 0

A binary addition

Unary $-(\cdot)$ function

$\Bigg\}$ "interpretations of the symbols"

$(\mathbb{Z}_3, 0, +, -)$ Universe $|\mathbb{Z}_3| = \{0, 1, 2\}$

$0^{\mathbb{Z}_3} = 0$

$+^{\mathbb{Z}} = \{ \langle 0,0,0\rangle, \langle 0,1,1\rangle, \langle 0,2,2\rangle, \langle 1,0,1\rangle, \langle 1,1,2\rangle, \langle 1,2,0\rangle \langle 2,0,2\rangle, \langle 2,1,0\rangle, \langle 2,2,1\rangle \}$

$-^{\mathbb{Z}} = \{ \langle 0,0\rangle, \langle 1,2\rangle, \langle 2,1\rangle \}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Axioms for groups

$$\forall x \forall y \forall z \, ((x+y)+z = x+(y+z))$$

$$\forall x \, (x+0=x \wedge 0+x=x)$$

$$\forall x \, (x+(-x)=0 \wedge (-x)+x=0).$$

Group := Any structure over language $\{0, +, -\}$ that satisfies these three axioms.

Definition of Truth

Structure $\mathfrak{A}$ — Gothic "A" — \mathfrak

$\mathfrak{B}$ — Gothic "B"

$\mathcal{M}, \mathcal{N}$ — "M = model"

$\mathcal{I}, \mathcal{J}$ — "I" = interpretation.

$a \quad \mathfrak{A} \quad \mathfrak{A}$

Formula A — $\mathfrak{A} \models A$

Definition of truth will use induction of formulas A

e.g. A is $\forall x_2 \exists x_1 (x_1 + x_1 = x_2)$

Subformula $\boxed{\exists x_1 (x_1 + x_1 = x_2)}$ also needs a definition of its truth.

x_2 is a free variable

We an object assignment to give values to free variables.

Defin Fix a structure $\mathfrak{A}$, it has a universe of objects, called $|\mathfrak{A}|$

An object assignment is a mapping

$$\sigma : \{x_1, x_2, \ldots\} \to |\mathfrak{A}|$$

We'll write $\mathfrak{A} \models A[\sigma]$ to mean A is true in $\mathfrak{A}$ with object assignment σ.

Fix a language L.

Definition: An L-structure or L-interpretation $\mathfrak{A}$ consists of

(1) A universe or domain - a non-empty set of objects or individuals, denoted $|\mathfrak{A}|$

(2) For each constant symbol $c \in L$, a object $c^{\mathfrak{A}} \in |\mathfrak{A}|$ called the interpretation of c.

(3) For each k-ary predicate symbol $P \in L$, a set of k-tuples $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^k$

i.e $P^{\mathfrak{A}} = \{ \langle a_1, \ldots, a_k \rangle : \cdots \} \subseteq |\mathfrak{A}|^k$

intuition: $\langle a_1 .. a_k \rangle \in P^{\mathfrak{A}}$ iff $P(a_1 .. a_k)$ is true in $\mathfrak{A}$.

(4) For each function symbol $f \in L$, f k-ary, a set of $(k+1)$-tuples $f^{\mathfrak{A}} \subseteq |\mathfrak{A}|^{k+1}$ that defines the graph of a total function.

For all $a_1 .. a_k \in |\mathfrak{A}|$, there is a unique $a_{k+1} \in |\mathfrak{A}|$ such that $\langle a_1 .. a_k, a_{k+1} \rangle \in f^{\mathfrak{A}}$

$|\mathfrak{A}|$ is closed under f^{Ω}, f^{Ω} is total and is a function.

for every $a_1, \ldots, a_k \in |\mathfrak{A}|$, exists a (unique) a_{k+1} s.t. $\langle a_1 \ldots a_k, a_{k+1} \rangle \in f^{\Omega}$

Intuition: $a_{k+1} = f(a_1 \ldots a_k)$.

Notation: $a_{k+1} = f^{\Omega}(a_1 \ldots a_k)$ will just mean $\langle a_1 \ldots a_k, a_{k+1} \rangle \in f^{\Omega}$.

f^{Ω} is the graph of a function.

Define truth

1. Denotation of terms
2. Truth of atomic formulas
3. " " formulas.

Defn Let $\mathfrak{A}$ be L-structure, σ an object assignment for $\mathfrak{A}$. Then, for t a L-term, the σ is extended to have domain the set of all L-terms as follows:

(1) For t a variable x_i, $\sigma(x_i)$ is already defined.

(2) For t a constant c, $\sigma(t) = c^{\mathfrak{A}}$

(3) For t of the form $f(t_1, .., t_k)$,

$$\sigma(t) = f^{\mathfrak{A}}(\sigma(t_1), \dots, \sigma(t_k)).$$

Example In $\mathbb{Z}_3$, if $\sigma(x_1) = 0$ and $\sigma(x_2) = 1$,

then $\sigma((x_1 + x_2)) = 1 = 0 +^{\mathfrak{A}} 1$

Definition Let A be an atomic L-formula. Then $\mathfrak{A} \models A[\sigma]$ is defined by:

(1) A is $P(t_1 .., t_k)$, then $\mathfrak{A} \models A[\sigma]$ if and only if $\langle \sigma(t_1), .., \sigma(t_k) \rangle \in P^{\mathfrak{A}}$.

(2) If A is $s = t$, then $\mathfrak{A} \models A[\sigma]$ iff $\sigma(s) = \sigma(t)$.

$\mathfrak{A}$ - L-structure; σ - an object assignment for $\mathfrak{A}$; A - L-formula.

Definition of truth: $\mathfrak{A} \models A[\sigma]$ "A is in $\mathfrak{A}$ with σ"

or "$\mathfrak{A}$ satisfies A with σ".

Model $\sigma(A) = T$ for $\mathfrak{A} \models A[\sigma]$

is defined inductively by:

(1) If A is atomic already defined (base case)

(2) If A is $\neg B$, then $\mathfrak{A} \models A[\sigma]$ if $\mathfrak{A} \not\models B[\sigma]$ $\quad \not\models$

(3) If A is $(B \wedge C)$ then $\mathfrak{A} \models A[\sigma]$ if $\mathfrak{A} \models B[\sigma]$ and $\mathfrak{A} \models C[\sigma]$

(4) If A is $(B \vee C)$ then $\mathfrak{A} \models A[\sigma]$ if $\mathfrak{A} \models B[\sigma]$ or $\mathfrak{A} \models C[\sigma]$

(5) If A is $(B \rightarrow C)$ then $\mathfrak{A} \models A[\sigma]$ if $\mathfrak{A} \not\models B[\sigma]$ or $\mathfrak{A} \models C[\sigma]$.

(6) If A is $(B \leftrightarrow C)$ then $\mathfrak{A} \models A[\sigma]$ if $\mathfrak{A} \models B[\sigma]$ and $\mathfrak{A} \models C[\sigma]$

both are true or both are false

(7) If A is $\forall x_i B$, then $\mathfrak{A} \models A[\sigma]$ if

For every x_i-variant τ of σ, $\mathfrak{A} \models B[\tau]$.

Defn An x_i-variant of σ is any object assignment τ

such that $\tau(x_j) = \sigma(x_j)$ for all $j \neq i$.

(8) If A is $\exists x_i B$, then $\mathfrak{A} \models A[\sigma]$ if, for some x_i-variant τ of σ,

$\mathfrak{A} \models B[\tau]$.

Comment: The truth of $\mathfrak{A} \models A[\sigma]$ only depends on the values $\sigma(x_i)$ for those x_i's that have free occurrences in A.

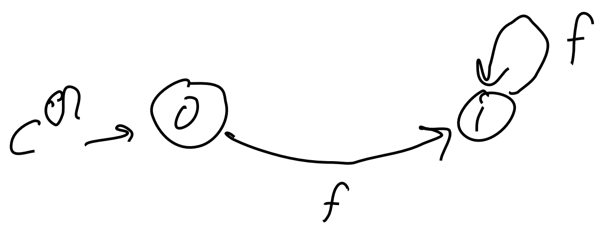
So if A is a sentence, then $\mathfrak{A} \models A[\sigma]$ does not depend on σ at all.

In this case, we write $\mathfrak{A} \models A$ (with no need to mention a σ.)

Examples:

Language $L = \{f, c\}$ f - unary function
c - constant symbol.

L-structure $\mathfrak{A}$: $|\mathfrak{A}| = \{0, 1\}$, $c^{\mathfrak{A}} = 0$, $f^{\mathfrak{A}} = \{\langle 0,1\rangle, \langle 1,1\rangle\}$



$\mathfrak{A} \models \exists x\, \forall y\, (f(y) = x)$, Then $\mathfrak{A} \models \forall y\, f(y) = x)\,[\tau]$

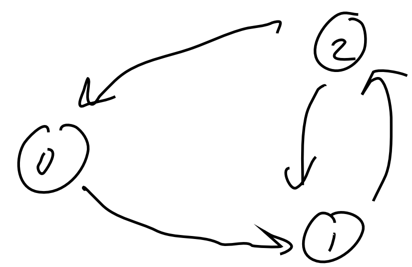
Let $\tau(x) = 1$ (Think: x is x_1, y is x_2)

Example: Language $L = \{E\}$ - binary relation.

Language of directed graphs with loop.

Structure $\mathcal{B}$ $\quad |\mathcal{B}| = \{0, 1, 2\}$

$E^{\mathcal{B}} = \{\langle 0,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,0\rangle\}$



$\mathcal{B} \models \forall x \exists y\, E(y,x)$. "every vertex x has an incoming edge".

$\mathcal{B} \models \exists y\, (E(y,x) \wedge E(x,y))[\sigma]$ if $\sigma(x) = 1$ or $\sigma(x) = 2$