Midterm

≥ 90 : 3	60's 7	Median 57.
80's : 6	50's 12	Mean: 60
70's 4	40's 9	
	<40 5	

Jums. ucsd.edu - Lean Theorem prover.

Completeness Theorem:

(a) If Γ is consistent, then Γ is satisfiable

(b) If $\Gamma \vDash A$, then $\Gamma \vdash A$

Showed already: (b) $\Rightarrow$ (a).

Lindenbaum's Theorem: If Γ is consistent, then there is a complete and consistent $\Pi \supseteq \Gamma$.

Defn Π is complete if $\forall A$, $A \in \Pi$ or $\neg A \in \Pi$.

(Could have used: either $\Pi \vdash A$ or $\Pi \vdash \neg A$)

Need to show There is a truth assignment φ that satisfies Π and hence Π.

Lemma: Suppose Π is consistent and complete.

(1) For any A, $A \in \Pi$ iff $\neg A \notin \Pi$

(2) For any A, B: $A \to B \in \Pi$ iff $A \notin \Pi$ or $B \in \Pi$

Pf: Proof of part (1). By ~~complete~~ consistency property, A and $\neg A$ are not both in Π.
By "complete" property, at least one of A, $\neg A$ is in Π. $\square$

Proof of (2):

Assume $A \to B \in \Pi$. Want to show $A \notin \Pi$ or $B \in \Pi$.

Assume also, $A \in \Pi$ and $B \notin \Pi$ (for sake of a contradiction)

By (1), $\neg B \in \Pi$. So $\{A \to B, A, \neg B\} \subseteq \Pi$ - but this is inconsistent #

Assume: $A \notin \Pi$ and $A \to B \notin \Pi$. (for sake of a contradiction)

By (1), $\neg(A \to B) \in \Pi$ Similarly, $\neg A \in \Pi$.

But $\{\neg A, \neg(A \to B)\}$ is inconsistent since $\neg A \vdash A \to B$

Assume $B \in \Pi$ and $A \to B \notin \Pi$. By (1), $\neg(A \to B) \in \Pi$. #

and $\{B, \neg(A \to B)\}$ is inconsistent since $B \vdash A \to B$

qed Lemma.

Given Π is consistent and complete.

Want to show: exists a φ satisfying Π.

$$\boxed{\text{Set } \varphi(p_i) = \begin{cases} T & \text{if } p_i \in \Pi \\ F & \text{if } p_i \notin \Pi \end{cases}}$$

Claim: $\varphi(A) = T$ iff $A \in \Pi$, for all formulas A.

Pf by on the complexity of A.

Base case: A is a propositional variable p_i

Claim holds in this case directly from the definition of φ

Ind Step #1: A is $\neg B$.

$\varphi(A) = T$	$\iff \varphi(B) = F$	by defn of truth
	$\iff B \notin \Pi$	by induction hypothesis
	$\iff A \in \Pi$	by Lemma (1) ✓

Ind Step #2: A is $(B \to C)$.

$\varphi(A) = T$	$\iff \varphi(B) = F$ or $\varphi(C) = T$	by defn of truth
	$\iff B \notin \Pi$ or $C \in \Pi$	by the two inductive hypotheses
	$\iff B \to C \in \Pi$	by Lemma (2)
	$\iff A \in \Pi$	by choice of A, B, C

☑ QED Completeness Theorem!!!

First-order logic:

Extend propositional logic

Keep $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

Discard $p_1, p_2, \ldots$

Add <u>variables</u> $x_1, x_2, x_3 \ldots$ $x, y, z, \ldots$ range of a domain universe of objects (aka individuals)

<u>Predicates</u> (aka, Relations) such as $\leq$,

take objects as inputs & return T or F values,

<u>Functions</u> such as $+, \circ$

Input to functions is individuals

Output is an individual.

<u>Constants</u> - Names for particular objects. 0

$\Longrightarrow$ <u>Quantifiers</u> $\forall, \exists$ $\forall x(\ldots)$ $\exists x(\ldots)$

Examples

John - constant

Dog(x) - "x is a dog"
Cat(x) - "x is a cat"
Person(x) - "x is a person"

} 1-ary (unary) predicates.

Likes(x,y) - "x likes y"

Mother(x) = mother of x (an individual)

$\forall x(\neg Dog(x) \vee \neg Cat(x))$ "No dog is a cat"

$\forall x(\neg(Dog(x) \wedge Cat(x)))$ " " " "

$\forall x \forall y(Dog(x) \wedge Cat(y) \rightarrow x \neq y)$

$a = b$ $\quad \neg(a=b)$ $\quad$ $a \neq b$ means $\neg(a=b)$

$=$ - special predicate symbol for "true" equality.

$\forall x(Likes(John, x) \rightarrow Cat(x))$ - John likes only cats.
John doesn't like anything except cats
- Each thing John likes is a cat.
forall "All"

"John likes all cats"

$\forall x\,(\text{Likes}(\text{John},x))$ ← John likes everything

$\forall x\,(\text{Cats}(x) \to \text{Likes}(\text{John},x))$

$\neg\exists x\,(\text{Cats}(x) \wedge \neg\text{Likes}(\text{John},x))$

$\neg\exists x\,\neg(\text{Cats}(x) \to \text{Likes}(\text{John},x))$

$\forall x A \quad \models\dashv \quad \neg\exists x\neg A \qquad\qquad \forall \quad \neg\exists\neg$

"John likes some cat"

$\exists x\,(\text{Cat}(x) \wedge \text{Likes}(\text{John},x))$

X

$\exists x\,(\text{Cat}(x) \to \text{Likes}(\text{John},x))$ - true as long there is some x that is not a cat or something John likes

$\exists x\,(\text{Cat}(x) \vee \text{Likes}(\text{John},x))$

For all reals x $(\cdots\cdots)$

$\forall x \in \mathbb{R} (\cdots)$ "bounded quantifier"

$\forall x (x \in \mathbb{R} \to \cdots)$

For some real x $(\cdots)$

$\exists x \in \mathbb{R} (\cdots)$

$\exists x (x \in \mathbb{R} \wedge \cdots)$