Soundness Theorem:

(a) If Γ is satisfiable, then Γ is consistent.

(b) If $\Gamma \vdash A$, then $\Gamma \models A$

Completeness theorem

(a) If Γ is consistent, then Γ is satisfiable.

(b) If $\Gamma \models A$, then $\Gamma \vdash A$.

Compactness

Defn Γ is finitely satisfiable if every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

Compactness Theorem

(a) Γ is satisfiable if and only if Γ is finitely satisfiable

(b) $\Gamma \models A$ if and only if $\exists$ finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \models A$.

Pf of Compactness, part (a):

"only if" $\Rightarrow$ - Easy

"if" $\Leftarrow$: Suppose Γ is unsatisfiable (& Γ is finitely satisfiable)

Then Γ is inconsistent by Completeness (a)

So there is a finite $\Gamma_0 \subseteq \Gamma$ s.t. Γ_0 is inconsistent

i.e. there is a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \vdash \neg(p_1 \to p_1)$

So by Soundness, $\exists$ finite $\Gamma_0 \subseteq \Gamma$ s.t. Γ_0 is unsatisfiable

Let's prove part (b) of Completeness from part (c)

Assume $\Gamma \nvdash A$. Want to show $\Gamma \nvDash A$.

By $\Gamma \nvdash A$, $\Gamma \cup \{\neg A\}$ is consistent.

By part (a) of Completeness, $\Gamma \cup \{\neg A\}$ is satisfiable.

So $\Gamma \nvDash A$.

qed.

Let's prove part (a) of the Soundness from part (b)

Assume Γ is inconsistent. Want to show Γ is unsatisfiable.

Since Γ is inconsistent, $\Gamma \vdash \neg(p_1 \to p_1)$.

By part (b), $\Gamma \vDash \neg(p_1 \to p_1)$

But $\varphi(\neg(p_1 \to p_1)) = F$ for all φ, so Γ is unsatisfiable.

qed.

Soundness part (b) If $\Gamma \vdash A$, then $\Gamma \models A$.

Assume $\Gamma \vdash A$

Let $B_1, B_2, \ldots, B_k$ be a PL-proof of A from Γ

- B_k is A.

We'll show $\Gamma \models B_i$ for $i = 1, \ldots, k$, by induction on i

Case (1): B_i is a PL-axiom or $B_i \in \Gamma$ (a hypothesis)

Then $\Gamma \models B_i$ (since all axioms are tautologies

Case (2) B_i is inferred by M.P. from B_j and B_k, $j, k < i$

with B_k in the form $B_j \rightarrow B_i$

Induction hypotheses: $\Gamma \models B_j$ ~~$\models B_j \rightarrow B_i$~~ $\Gamma \vdash B_j \rightarrow B_i$

Let φ be any truth assignment satisfying Γ,

We need to show $\varphi(B_i) = T$.

We know $\varphi(B_j) = T$ and $\varphi(B_j \rightarrow B_i) = T$.

By definition of truth for $\rightarrow$, $\varphi(B_i) = T$.

q.e.d.

Proof of part (a) of the Completeness: Γ is consistent $\Rightarrow$ Γ is satisfiable

Definition: Γ is complete if, for all A, either $A \in \Gamma$ or $\neg A \in \Gamma$.

Lindenbaum's Theorem. Suppose Γ is consistent.
Then there is a consistent and complete $\Pi \supseteq \Gamma$.

Example $\Gamma = \{p_1\}$, $\Pi = \{p_1, p_2, p_3, \dots\} \cup \{$all tautological consequences of $\{p_1, p_2, \dots\}\}$

Proof of Lindenbaum's Theorem:

Step (1) Enumerate all formulas ($\{\neg, \rightarrow\}$-formulas) as a countable list
$A_1, A_2, A_3, \dots$

E.g For $k = 1, 2, 3, \dots$ loop:
 Enumerate all formulas with $\leq k$ symbols
 using only variables $p_1, \dots, p_k$
end for.

Step (2) Define $\Gamma_0 = \Gamma$

$$\Gamma_i = \begin{cases} \Gamma_{i-1} \cup \{A_i\} & \text{if this is consistent} \\ \Gamma_{i-1} \cup \{\neg A_i\} & \text{otherwise} \end{cases}$$

Let $\Pi = \bigcup_i \Gamma_i$

$\Gamma_0 = \Gamma$

$$\Gamma_i = \begin{cases} \Gamma_{i-1} \cup \{A_i\} & \text{if consistent} \\ \Gamma_{i-1} \cup \{\neg A_i\} & \text{otherwise} \end{cases}$$

$\Pi = \bigcup_i \Gamma_i$

"limit of the Γ_i's"

☆ <u>Claim</u> For all A, either $A \in \Pi$ or $\neg A \in \Pi$.

<u>Pf</u> This is clear – since every A is an A_i

<u>Claim</u> Γ_i is consistent for all i

<u>Claim</u> $\Gamma_i \supseteq \Gamma_{i-1}$

<u>Handout</u>: If Γ_{i-1} is consistent, then at least one of $\Gamma_{i-1} \cup \{A_i\}$ or $\Gamma_{i-1} \cup \{\neg A_i\}$ is consistent

Therefore, by induction on i, each Γ_i is consistent.

☆ <u>Claim</u> Π is consistent.

This is proved by the argument on the handout.

So Π is complete & consistent. Also $\Pi \supseteq \Gamma$ (Since $\Gamma_0 = \Gamma$, $\Gamma_0 \subseteq \Gamma_i \; \forall i$)

QED. (Lindenbaum's theorem)