Def'n Γ is inconsistent if $\Gamma \vdash A$ and $\Gamma \vdash \neg A$, for some A

Example $\{A \to B, A, \neg B\}$ is inconsistent.

Theorem 1 If Γ is inconsistent and $\Pi \supseteq \Gamma$, then Π is inconsistent.

Proof $\boxed{}$

Theorem 2 If Γ is inconsistent, then some finite subset Γ_0 of Γ is inconsistent.

Theorem 3: If $\Gamma \vdash A$, then $\Gamma_0 \vdash A$ for some finite $\Gamma_0 \subseteq \Gamma$

Pf of Thm 3: Proofs are finite - only finitely many hypotheses can be used in a proof. $\boxed{}$

Theorem 2 immediate from theorem 3. $\boxed{}$

Theorem 4: If Γ is inconsistent, then $\Gamma \vdash B$ for every B.

Pf: Suppose $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. Want to show $\Gamma \vdash B$.

PL3 axiom: $\neg A \to A \to B$ - Use MP twice

"Modus Ponens

Theorem (Proof by Contradiction #1):

$\Gamma \vdash A$ if and only if $\Gamma \cup \{\neg A\}$ is inconsistent.

Proof: "only if" Suppose $\Gamma \vdash A$. Then $\Gamma \cup \{\neg A\} \vdash A$

Thus $\Gamma \cup \{\neg A\}$ is inconsistent (Since $\Gamma, \neg A \vdash \neg A$)

"if": Suppose $\Gamma \cup \{\neg A\}$ is inconsistent.

Thus $\Gamma \cup \{\neg A\} \vdash A$ (by Theorem 4)

By Deduction Theorem, $\Gamma \vdash \neg A \to A$.

Axiom PL4 $(\neg A \to A) \to A$. ~~So~~

So by MP, $\Gamma \vdash A$ $\boxed{\text{qed}}$

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Corollary $\neg\neg A \vdash A$ and $\vdash \neg\neg A \to A$

Pf: Suffices to show $\{\neg\neg A, \neg A\}$ is inconsistent

That is obvious.

qed

Theorem (Proof by Contradiction #2)

$\Gamma \vdash \neg A$ if and only if $\Gamma \cup \{A\}$ is inconsistent.

Pf "only if" Suppose $\Gamma \vdash \neg A$

Then $\Gamma \cup \{A\} \vdash \neg A$.

So $\Gamma \cup \{A\}$ is inconsistent.

"if" Suppose $\Gamma \cup \{A\}$ is inconsistent.

So $\Gamma, A \vdash \neg A$. Therefore $\Gamma \vdash A \to \neg A$.

By the Corollary, $\Gamma \vdash \neg\neg A \to A$.

By Hypothetical Syllogism, $\Gamma \vdash \neg\neg A \to \neg A$

By PL3 $(\neg\neg A \to \neg A) \to \neg A$, and MP,

$\Gamma \vdash \neg A$.

q.e.d. $\boxed{\checkmark}$

Corollary: $A \vdash \neg\neg A$ and $\vdash A \to \neg\neg A$.

Pf. By Theorem, suffices to show $\{A, \neg A\}$ is inconsistent. And it is!

qed $\boxed{\checkmark}$

Modus Tollens – derived rule of inference

$$\frac{A \to B \qquad \neg B}{\neg A}$$

Theorem $\vdash (A \to B) \to \neg B \to \neg A$

Equivalently $\boxed{A \to B, \neg B \vdash \neg A}$

~~Pf: Suffices to $\{A \to B, \neg B, \neg A\}$ is inconsistent~~
~~(Pf by Contradiction #2)~~

~~thus, suffices to show $A \to B$~~

Proof Suffices to show $\{A \to B, \neg B, A\}$ is inconsistent
(By Pf by Contradiction #2

Thus, suffices to show: $A \to B, A \vdash B$
This holds by one use of Modus Ponens.

qed

Proof by Cases:

Theorem • If $\Gamma \vdash A \to B$ and $\Gamma \vdash \neg A \to B$,
then $\Gamma \vdash B$

• $A \to B,\ \neg A \to B \vdash B$

• $\vdash (A \to B) \to (\neg A \to B) \to B$.

Proof of $A \to B,\ \neg A \to B \vdash B$

Let's use Modus Tollens. + Proof by Contradiction.

Suffices to show $\{A \to B,\ \neg A \to B,\ \neg B\}$ is inconsistent.

By Modus Tollens twice,

$$\{\underline{A \to B}, \neg A \to B, \neg B\} \vdash \neg A$$

and $\{A \to B,\ \underline{\neg A \to B}, \neg B\} \vdash \neg\neg A$

$$\frac{A \to B,\ \neg B}{\neg A} \qquad \frac{\neg A \to B \quad \neg B}{\neg\neg A}$$

$\square$

Last step could instead be

$\{A \to B,\ \neg A \to B,\ \neg B\} \vdash B$ by MP.

Upcoming Big Theorems

Soundness Theorem

(a) If Γ is satisfiable, then Γ is consistent.

(b) If $\Gamma \vdash A$, then $\Gamma \models A$.

Completeness Theorem

★!

(a) If Γ is consistent, then Γ is satisfiable

(b) If $\Gamma \models A$, then $\Gamma \vdash A$

Compactness Theorem

(a) Γ is satisficble if and only if every finite subset of Γ is satisfiable.

(b) $\Gamma \models A$ if and only if for some finite subset Γ_0 of Γ,

$$\Gamma_0 \models A$$

Proof of Compactness is easy from Soundness & Completeness

I.e. • Γ is consistent iff every finite $\Gamma_0 \subseteq \Gamma$ is consistent.

• $\Gamma \vdash A$ iff, for some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vdash A$.