Review session, in-person, next week, Wed @ 7:00?

Hilbert-style proof system PL

PL - works only with $\{\neg, \to\}$-formulas.

<u>Henceforth</u> $A \vee B$ is an abbreviation for $\neg A \to B$

$A \wedge B$ " " " " $\neg(A \to \neg B)$

$A \leftrightarrow B$ " " " " $(A \to B) \wedge (B \to A)$

<u>PL</u> 4 axiom schemes

Every substitution instance of the formulas:

$A \to B \to A$

PL 1 : $p_1 \to p_2 \to p_1$

PL 2 : $(p_1 \to p_2 \to p_3) \to (p_1 \to p_2) \to (p_1 \to p_3)$

$(A \to B \to C) \to (A \to B) \to (A \to C)$

PL 3: $\neg p_1 \to p_1 \to p_2$ $\qquad \neg A \to A \to B$

PL 4: $(\neg p_1 \to p_1) \to p_1$ $\qquad (\neg A \to A) \to A$

There are tautologies.

PL - one rule of inference - Modus Ponens $\dfrac{A \quad A \to B}{B}$

Def'n A PL-proof (PL-derivation) of A is sequence of formulas $B_1, B_2, \ldots, B_\ell$ where

(1) B_ℓ is A "the conclusion of the proof.

(2) Each B_i is either a PL-axiom, or

is inferred by Modus Ponens from B_j, B_k $j, k < i$

That is B_k is $B_j \to B_i$ $\dfrac{B_j \quad B_j \to B_i}{B_i}$

We say A is a theorem (PL-theorem)

Notation $\vdash A$ - "A has a PL-proof". \vdash

Example $\vdash p_1 \to (p_2 \vee p_3) \to p_1$ PL1-axiom

has one line proof!

<u>Example</u> $\vdash (\neg A \to A) \to (\neg A \to B)$ for all A, B.

<u>PL proof</u>: $\neg A \to A \to B$ PL3 axiom

$(\neg A \to A \to B) \to (\neg A \to A) \to (\neg A \to B)$ PL2 axiom

$(\neg A \to A) \to (\neg A \to B)$ Modus Ponens

<u>Example</u>: $\vdash B \to A \vee B$

i.e. $\vdash B \to \neg A \to B$ PL1 axiom.

(Later: $\vdash A \to A \vee B$)

<u>Example</u> $\vdash \neg\neg A \to A \vee B$

i.e $\vdash \neg\neg A \to \neg A \to B$ PL3 axiom.

<u>Example</u>: $\vdash A \to A$ for all formulas A.

1. $(A \to (A \to A) \to A) \to (A \to (A \to A)) \to (A \to A)$ PL2 axiom
2. $A \to (A \to A) \to A$ PL1 axiom
3. $(A \to (A \to A)) \to (A \to A)$ Modus Ponens; 1, 2,
4. $A \to (A \to A)$ PL1 axiom
5. $A \to A$ Modus Ponens

<u>Def'n</u> Let Γ be a set of formulas. A <u>proof</u> of A <u>from the hypotheses</u> Γ is a sequence of formulas

$$B_1, \ldots, B_\ell$$

such that

① B_ℓ is A

② Each B_i is

ⓐ A PL-axiom, or

ⓑ A member of Γ $\quad (B_i \in \Gamma)$, or $\quad$ New!

ⓒ B_i is inferred by Modus Ponens from $B_j, B_k; j, k < i$

i.e. B_k is $B_j \to B_i$

We say "A is a <u>theorem</u> of Γ", notation $\Gamma \vdash A$

Members of Γ are called "hypotheses"

or "non-logical axioms"

PL axioms - sometimes called "logical axioms"

<u>Theorem</u> If $\Gamma \vdash A$ and $\Pi \supseteq \Gamma$, then $\Pi \vdash A$.

"obvious":

<u>Example</u> $\{A \to B, A\} \vdash B$ <u>i.e</u> $A \to B, A \vdash B$

<u>PL proof</u>

	A	hypothesis (HYP)
	$A \to B$	"
	B	Modus Ponens

<u>Deduction Theorem:</u>

(a) $A \vdash B$ if and only $\vdash A \to B$.

(b) $T, A \vdash B$ if and only $T \vdash A \to B$.

(a) is a special case of (b), namely $T = \emptyset$.

<u>Pf of (b):</u> "if". Suppose $T \vdash A \to B$

So $T, A \vdash A \to B$

Also $T, A \vdash A$ "HYP"

By Modus Ponens, $T, A \vdash B$.

Proof ⑥ continued "only if"

Suppose $T, A \vdash B$.

Let $C_1, C_2, \ldots, C_\ell$ be a proof of B from $T \cup \{A\}$.

So C_ℓ is B.

We prove $\boxed{T \vdash A \to C_i}$ for all i, by induction on i.

Case (1): C_i is a PL-axiom or a member of T

So $T \vdash C_i$

By the PL-axiom $C_i \to A \to C_i$ and Modus Ponens

$T \vdash A \to C_i$

Case 2: C_i is A. $\vdash A \to A$ by earlier example.

So $T \vdash A \to C_i$

Case 3 C_i is inferred by Modus Ponens from C_j, C_k; C_k is $C_j \to C_i$

2 induction hypotheses: $T \vdash A \to C_j$ and $T \vdash A \to (C_j \to C_i)$

PL2 axiom: $(A \to C_j \to C_i) \to (A \to C_j) \to (A \to C_i)$

So by 2 uses of Modus Ponens $T \vdash A \to C_i$. ☐

Example $\vdash A \to (A \vee B)$ i.e. $\vdash A \to (\neg A \to B)$

$\vdash A \to (\neg A \to B)$

$\iff A \vdash \neg A \to B$ by Deduction Theorem

$\iff A, \neg A \vdash B$ " " "

By PL3, $\neg A \to A \to B$ is an axiom.

$A, \neg A \vdash \neg A \to A \to B$ PL3

$A, \neg A \vdash \neg A$ Hyp

$A, \neg A \vdash A \to B$ By Modus Ponens

$A, \neg A \vdash A$ Hyp

$A, \neg A \vdash B$ By Modus Ponens.

So $A, \neg A \vdash B$. $\boxed{\checkmark}$

Theorem: $A, \neg A \vdash B$ for all A, B.

Hypothetical Syllogism

$$\frac{A \to B \qquad B \to C}{A \to C}$$

Derived rule of inference

Admissible rule of inference

Theorem $\boxed{A \to B,\ B \to C \vdash A \to C}$

Equivalently: $\boxed{\vdash (A \to B) \to (B \to C) \to (A \to C)}$ (by 2 uses of the Deduction Theorem)

So If $\Gamma \vdash A \to B$ and $\Gamma \to (B \to C)$
then $\Gamma \vdash A \to C$.

Proof the Theorem:

$A \to B,\ B \to C \vdash A \to C$

$\iff \boxed{A \to B,\ B \to C,\ A \vdash C}$ by Deduction Theorem.

~~By 2 use~~ $A \to B, B \to C, A \vdash B$ by 2 uses of Hyp and Modus Ponens

So $\boxed{A \to B,\ B \to C,\ A \vdash C}$ by a Hyp and Modus Ponens.

$\square$