"p ~~and~~ or q but not both"

$(p \vee q) \wedge (\neg p \vee \neg q)$ - CNF - conjunctive normal form "And"

$(p \wedge \neg q) \vee (\neg p \wedge q)$ - DNF - disjunctive normal form. "or"

$(p \leftrightarrow \neg q)$ or $(\neg p \leftrightarrow q)$ or $\neg(p \leftrightarrow q)$.

<u>Notation</u>: $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$-formula means ~~our~~ our usual formula.

$\{\neg, \leftrightarrow\}$-formulas

<u>Defn</u> Let L be a set of connectives, an L-formula is a formula using only connectives from L.

erroneously

<u>Quiz</u> erroneously asked about $\{\wedge, \leftrightarrow\}$-formulas.

<u>Theorem</u> Let A be $\{\wedge, \vee, \leftrightarrow, \rightarrow\}$ formula.
If $\varphi(p_i) = T$ for all i, then $\varphi(A) = T$

<u>Corollary</u> $\neg p$ and "p or q but not both" cannot be expressed as a $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$ formula.

Proof Use induction ~~on~~ on the complexity of a $\{\wedge, \vee, \leftrightarrow, \rightarrow\}$-formula.

Suppose $\varphi(p_i) = T$ for all i.

Base case: A is a variable p_i

By assumption, $\varphi(A) = T$

Induction steps: A is $(B \circ C)$ where $\circ$ is $\wedge, \vee, \rightarrow$ or $\leftrightarrow$

Induction hypotheses: $\varphi(B) = T$ and $\varphi(C) = T$.

By the definition of truth for $\circ$,

$\varphi(A) = T$.

qed.

Question: Does $\neg, \wedge, \vee, \leftrightarrow, \rightarrow$ suffice to define/express everything we want?

Answer: Yes.

Purpose of a formula is to express a Boolean function.

Defin A k-ary Boolean function f is a mapping

$$f: \{T, F\}^k \to \{T, F\}$$

$$f(x_1, \ldots x_k) \in \{T, F\} \quad \text{if } x_1, \ldots, x_k \in \{T, F\}$$

"Generalized connective"

Defin Let A be a propositional formula using only variables $p_1, \ldots, p_k$, then f_A is the Boolean function defined by A:

$$f_A(x_1, \ldots x_k) = \varphi(A) \quad \text{if} \quad \varphi(p_i) = x_i \text{ for all } i.$$

Defin Let L be a set of connectives ("L" = "language")

L is adequate if every Boolean function is defined by some L-formula.

Also known as "truth functionally complete"

Theorem $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is adequate.

Example: $f_{p_2 \rightarrow p_1}$

x_1	x_2	$f(x_1, x_2)$	
T	T	T	1
T	F	T	2
F	T	F	
F	F	T	4

$f_{p_2 \rightarrow p_1}(x_1, x_2) = T$

iff

$(x_1 = T \ \& \ x_2 = T)$ or $(x_1 = T$ and $x_2 = F)$ or $(x_1 = F$ and $x_2 = F)$

Our formula A is $(p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge \neg p_2)$

Now $\neg p_2 \vee p_1$ already works

$\mathcal{I} = \{1, 2, 4\}$

$C_1 := p_1 \wedge p_2 \qquad C_2 := p_1 \wedge \neg p_2 \qquad C_3 := \neg p_1 \wedge \neg p_2$

Proof (1) $f(x_1, \dots x_k) = F$ for all $x_1, \dots, x_k \in \{T, F\}$

Use A is $\neg p_1 \wedge p_1$

(2) Otherwise. There 2^k many possible inputs to f.

Each corresponds to a truth assignments

Call these $\varphi_1, \dots, \varphi_{2^k}$ (the order doesn't matter)

Let L_{ij} be $\begin{cases} p_j & \text{if } \varphi_i(p_j) = T \\ \neg p_j & \text{if } \varphi_i(p_j) = F \end{cases}$ L_{ij} is a "literal"

So $\varphi_i(L_{ij}) = T$.

Set $C_i = L_{i1} \wedge L_{i2} \wedge \dots \wedge L_{ik} = \bigwedge_{j=1}^{k} L_{ij}$

Thus $\varphi_i(C_i) = T$

$\varphi_i(C_{i'}) = F$ if $i \neq i'$

Set $\mathscr{I} = \{i_1, \dots i_\ell\}$ to the set of values i such that

$f(\varphi_i(p_1), \dots, \varphi_i(p_k)) = T$ Note $\mathscr{I} \neq \emptyset$.)

Set $A = \bigvee_{i \in \mathscr{I}} C_i = C_{i_1} \vee C_{i_2} \vee \dots \vee C_{i_\ell}$.

Corollary $\{\neg, \vee, \wedge\}$ is adequate.

(Corollary follows the proof.)

The formula is a disjunctive normal form (DNF) formula

Defn:
- A literal is a variable p_i or a negated variable $\neg p_i$
- A conjunction of literals is of the form

$$C := l_1 \wedge \cdots \wedge l_k \quad \text{for } l_i\text{'s literals, } k \geq 1$$

- A DNF formula is a disjunction

$$C_1 \vee \cdots \vee C_k \qquad k \geq 1$$

with each C_i a conjunction of ~~clause~~ literals.

Corollary Every Boolean function is defined by a DNF formula.

Corollary Every propositional formula is tautologically equivalent to a DNF formula.

Since $A \models\!\dashv B$ iff $f_A = f_B$

Theorem
- $\{\neg, \wedge\}$ is adequate
- $\{\neg, \vee\}$ is adequate
- $\{\neg, \rightarrow\}$ is adequate.

Pf.
- $p \vee q \models\!\dashv \neg(\neg p \wedge \neg q)$ De Morgan's Law
- $p \wedge q \models\!\dashv \neg(\neg p \vee \neg q)$ " " "
- $p \vee q \models\!\dashv \neg p \rightarrow q$

Definition A conjunctive normal formula (CNF formula):

- A disjunction of literals also known as a clause is of the form

$$C = l_1 \vee \cdots \vee l_k \qquad l_i\text{'s literals, } k \geq 1$$

- A CNF formula is a conjunction of clauses

$$C_1 \wedge C_2 \wedge \cdots C_k \qquad k \geq 1, \text{ each } C_i \text{ a clause.}$$

Theorem Any Boolean function can be defined by a CNF formula.

Proof: (Sketch)

Let $f^{\neg}(x_1, \ldots, x_k) = \begin{cases} T & \text{if } f(x_1, \ldots x_k) = F \\ F & \text{if } f(x_1 \ldots x_k) = T \end{cases}$

$f^{\neg}$ has a DNF formula defining it.

Let that be A

Then $\neg A$ defines f

$\neg(C_1 \vee \cdots \vee C_\ell) \models\!\dashv \neg C_1 \wedge \neg C_2 \wedge \cdots \wedge \neg C_\ell$,

$\neg C_i$ is $\neg(l_1 \wedge \cdots \wedge l_m) \models\!\dashv \neg l_1 \vee \cdots \vee \neg l_m$.

$\neg l_j$ is $\neg p_j$ or $\neg\neg p_j$

So $\neg A$ is tautologically equivalent to a CNF formula.

<u>Nand</u> (Sheffer stroke) "Not - and"

$\varphi(A|B) := \varphi(\neg(A \wedge B))$

<u>Nor</u> $\downarrow$

$\varphi(A \downarrow B) := \varphi(\neg(A \vee B))$

Thm $\{|\}$ is adequate

$A|A \models\!\dashv \neg A$

$A \wedge B \models\!\dashv \neg(A|B) \models\!\dashv (A|B)|(A|B)$

<u>Thm</u> $\{\downarrow\}$ is adequate

See in-class handout

$\varphi(\mathrm{Id}(A)) = \varphi(A)$ "identity connective"