Recall A is a tautology, $\vDash A$, $\quad \forall \varphi \, (\varphi(A) = T)$.

Γ is *satisfiable* if $\quad \exists \varphi \, (\varphi(B) = T \text{ for all } B \in \Gamma)$

"φ satisfies Γ"

$\Gamma \vDash A$, Γ *tautologically implies* A, if

$\forall \varphi \, (\text{if } \varphi \text{ satisfies } \Gamma, \text{ then } \varphi(A) = T)$

Notation: $A \vDash B$ means $\{A\} \vDash B$

$\Gamma, A \vDash B$ means $\Gamma \cup \{A\} \vDash B$

Sometimes ~~write~~ write $\Gamma \vDash \Delta$ $\qquad \Gamma, \Delta$ - sets of formulas

to mean $\forall B \in \Delta \, (\Gamma \vDash B)$

Defn A is *tautologically equivalent* to B if

$A \vDash B$ and $B \vDash A$.

In other words, $\forall \varphi \, (\varphi(A) = \varphi(B))$.

Notation: $A \vDash\dashv B$ $\qquad$ \vDash \Dashv

Theorem $A \models B$ if and only $\models A \to B$

Pf: $A \models B \iff \forall \varphi$ (if $\varphi(A) = T$, then $\varphi(B) = T$)

$\iff \forall \varphi \; (\varphi(A) = F$ or $\varphi(B) = T)$

$\iff \forall \varphi \; (\varphi(A \to B) = T)$ by defn of truth

$\iff \; \models A \to B$ $\boxed{\checkmark}$

Theorem $A \models\!\!\dashv B$ if and only if $\models A \leftrightarrow B$

Example $A \to B, \neg B \models \neg A$ ("Modus Tollens")

Pf Suppose (for contradiction) that for some φ

$\varphi(A \to B) = T$, $\varphi(\neg B) = T$ and $\varphi(\neg A) = F$

Then $\varphi(A) = T$ and $\varphi(B) = F$ by def'n of truth

Thus $\varphi(A \to B) = F$ by def'n of truth

\# Contradiction.

Example $\neg B, A \models \neg(A \to B)$

Theorem: For all formulas A and B, $\neg A, A \models B$

Pf $\{\neg A, A\}$ is unsatisfiable.

So no φ satisfies $\{\neg A, A\}$.

So Theorem "vacuously"

Theorem: $\Gamma \models A$ if and only if $\Gamma \cup \{\neg A\}$ is unsatisfiable

LHS $\iff \forall \varphi$ (if φ satisfies Γ, then $\varphi(A) = T$)

$\iff \forall \varphi$ (φ does not satisfy Γ or $\varphi(A) = T$)

$\iff \forall \varphi$ (φ does not satisfy Γ or $\varphi(\neg A) = F$).

$\iff$ RHS

"principle of Contradiction"

Corollary (a) $\neg A$ is a tautology if and only if $\{A\}$ is unsatisfiable

(b) A is a tautology if and only if $\{\neg A\}$ is unsatisfiable

Note $A \models\dashv \neg\neg A$

Part (b): Theorem with $\Gamma = \emptyset$ (empty set).

$A \to B \models\dashv \neg A \vee B$.

Γ, Δ, Π - sets of formulas

Compact notation for truth tables $p \to q \to r \models\!\dashv (p \wedge q) \to r$

p	q	r	$p \to (q \to r)$		$(p \wedge q) \to r$	
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

same

So
$p \to (q \to r)$
$\models\!\dashv p \wedge q \to r$

Reduced truth table

p	q	r	$p \to (q \to r)$		$(p \wedge q) \to r$	
F	-	-	T		F	T
T	F	-	T	T	F	T
T	T	T	T	T	T	T
T	T	F	F	F	T	F

same again

"-" don't care

Section II.6 of PDF

Hodel book - see in Exercises.

Sample Tautological Equivalences

$$\neg\neg A \models\!\dashv A$$

De Morgan's Laws: $\neg(A \vee B) \models\!\dashv \neg A \wedge \neg B$

$\neg(A \wedge B) \models\!\dashv \neg A \vee \neg B$

Distributivity: $A \wedge (B \vee C) \models\!\dashv (A \wedge B) \vee (A \wedge C)$

$A \vee (B \wedge C) \models\!\dashv (A \vee B) \wedge (A \vee C)$

<u>Example</u> (An infinite set Γ of formulas.)

Let $\Gamma = \{p_1 \leftrightarrow \neg p_2,\ p_2 \leftrightarrow \neg p_3,\ p_3 \leftrightarrow \neg p_4, \ldots\} = \{p_i \leftrightarrow \neg p_{i+1} : i \geq 1\}$

Q: Is Γ satisfiable? What are its satisfying assignments.

Yes!

Satisfying assignment #1

$\varphi_1(p_1) = T,\ \varphi_1(p_2) = F,\ \varphi_1(p_3) = T, \ldots$

$\varphi(p_{2i+1}) = T,\ \varphi(p_{2i}) = F \quad (\forall i)$

Satisfying assignment #2: Alternate $F, T, F, T \ldots$

$\varphi_2(p_{2i+1}) = F$ and $\varphi_2(p_{2i}) = T. \quad (\forall i)$

$\Gamma \models p_1 \leftrightarrow p_3$ ✓

$\Gamma \not\models p_1 \qquad \Gamma \not\models \neg p_1$

Is $\Gamma \cup \{p_1\}$ satisfiable? Yes φ_1

Is $\Gamma \cup \{\neg p_1\}$ satisfiable? Yes φ_2

~~\nvDash~~ \nvDash