Definition: The propositional formulas are inductively defined by

(1) p_i is a formula. "variable"

(2) If A is a formula, so is $\neg A$

(3) If A and B are formulas, then so are

$$(A \vee B), \quad (A \wedge B), \quad (A \to B), \quad (A \leftrightarrow B)$$

Inductive definition.

Basis for proofs by induction & definitions by recursion

A formula is an expression - a string of symbols - over the alphabet: $\wedge, \vee, \neg, \to, \leftrightarrow, (,), p_1, p_2, \ldots$

Examples $\quad p_3 \qquad \neg p_3 \qquad (\neg\neg p_3 \wedge p_4)$

$$((\neg p_3 \wedge p_2) \leftrightarrow p_5)$$

Not formulas (p_1) - extra parantheses

$\forall p_1 (p_1 \to p_1)$ - $\forall$ - not allowed.

Informal conventions - to write formulas more readably

(1) Omit parentheses to make more readable

Default precedence of operations

(a) $\neg$ is highest precedence

(b) $\wedge, \vee$ - next highest precedence

(c) $\to, \leftrightarrow$ - lowest precedence

Within one precedence - associate from right-to-left.

$p_1 \to p_2 \to p_3$ means $(p_1 \to (p_2 \to p_3))$

(has same truth values as $((p_1 \wedge p_2) \to p_3)$

$p_1 \wedge p_2 \to p_3$

$\wedge, \vee, \leftrightarrow$ - are associative in terms of truth values

(2) Often use $p, q, r \ldots$ instead of $p_1, p_2, p_3, \ldots$

Definition: A truth assignment φ is a mapping

$$\varphi : \{p_1, p_2, p_3, \ldots\} \to \{T, F\}$$

T = "True"
F = "False"

Definition: Let φ be a truth assignment. Extend φ to have domain the set of all formulas by:

(1) $\varphi(\neg A) = \begin{cases} T & \text{if } \varphi(A) = F \\ F & \text{if } \varphi(A) = T \end{cases}$

(2) $\varphi((A \wedge B)) = \begin{cases} T & \text{if } \varphi(A) = \varphi(B) = T \\ F & \text{otherwise} \end{cases}$

(3) $\varphi((A \vee B)) = \begin{cases} T & \text{if } \varphi(A) = T \text{ or } \varphi(B) = T \\ F & \text{otherwise} \end{cases}$

(4) $\varphi((A \to B)) = \begin{cases} T & \text{if } \varphi(A) = F \text{ or } \varphi(B) = T \\ F & \text{otherwise} \end{cases}$

(5) $\varphi((A \leftrightarrow B)) = \begin{cases} T & \text{if } \varphi(A) = \varphi(B) \\ F & \text{otherwise} \end{cases}$

Unique Readability
Any formula C is uniquely expressible as (0) A variable or (1) as $\neg A$ or (2)-(5) as $(A \circ B)$ where $\circ$ is one of $\wedge, \vee, \to, \leftrightarrow$

A definition by recursion - based on inductive definition of formulas.

- Uses Unique Readability - for any formula (not a variable), exactly one of (1)-(5) can hold.

Example: $((p \to q) \to (q \to p))$

4 possibilities for $\varphi(p)$, $\varphi(q)$

p	q	$(p \to q)$	$(q \to p)$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

If $\varphi(p) = F$ and $\varphi(q) = T$, then $\varphi((p \to q) \to (q \to p)) = F$

This $(p \to q) \to (q \to p)$ is satisfiable

but not a tautology (not tautologically valid)

Defn A is satisfiable if $\varphi(A) = T$ for some φ.

A is a tautology if $\varphi(A) = T$ for all φ.

Defn φ satisfies A if $\varphi(A) = T$.

Notation: $\vDash A$ means "A is a tautology"

"double turnstile" \vDash

Definitions: Let Γ be a set of formulas.

φ satisfies Γ if $\varphi(A) = T$ for all A in Γ.

and then Γ is satisfiable

$\Gamma \vDash A$, Γ tautologically implies A,

if for every φ, if φ satisfies Γ,

then $\varphi(A) = T$.

$A \vDash B$ means same as $\{A\} \vDash B$

$\Gamma, A \vDash B$ " " " $\Gamma \cup \{A\} \vDash B$

Γ is unsatisfiable if it is not satisfiable.

$\models A$, A is tautologically valid.

This means the same as $\emptyset \models A$

The empty set tautologically implies A.

$\varphi \models \Gamma$

<u>Example</u> $A \models A$ if $\varphi(A) = T$, then $\varphi(A) = T$.

If $A \in \Gamma$, then $\Gamma \models A$. (Trivial from definition)

<u>Example</u> $\{p, p \to q\} \models q$ ("Modus Ponens")

If $\varphi(p) = T$ and $\varphi(p \to q) = T$ then $\varphi(q) = T$.

$$\frac{A \quad A \to B}{B}$$

<u>Theorem</u> If $\Gamma \subseteq \Delta$ and $\Gamma \models A$, then $\Delta \models A$.

<u>Pf</u> Any φ that satisfies Δ also satisfies Γ.

<u>Next time</u>: <u>Theorem</u> $A \models B$ iff $\models A \to B$