Math 160A - Fall 2021 - Class Work - In Lecture, September 22
(Hand in for the instructor to upload to Gradescope)

Name: ______________________

PID: ______________________

1. Which of the following are true statements? (Mark "T" or "F".)

T (a) Cats have wings only if dogs have wings.

F (b) Parrots have wings only if dogs have wings.

T (c) Parrots have wings if dogs have wings.

T (d) If dogs have wings then parrots have wings.

T (e) If cats have wings and dogs have wings, then cats have wings or dogs have wings.

T (f) If cats have wings or dogs have wings, then cats have wings and dogs have wings.

2. Rewrite items (a)-(c) above as "if $\cdots$ then $\cdots$" statements.

"If cats have wings, then dogs have wings."
"If parrots have wings, then dogs have wings."
"If dogs have wings, then parrots have wings."

3. Let $\mathbb{Z}$ denote the set of integers. Describe, as simply as possible, the following sets in English:

(a) $\{2k : k \in \mathbb{Z}\}$ The set of even integers

(b) $\{4n : n \in \mathbb{Z}\} \cup \{4n + 2 : n \in \mathbb{Z}\}$ The set of even integers

(c) $\{n : n = 2k \text{ for some } k \in \mathbb{Z}\}$ The set of even integers

(d) $\{n : n = 2k \text{ for all } k \in \mathbb{Z}\}$ The empty set. ($\emptyset$)

(e) $\{4k : k \in \mathbb{Z}\} \setminus \{2k : k \in \mathbb{Z}\}$ The empty set. ($\emptyset$)

Your comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, September 28
(Hand in for the instructor to upload to Gradescope)

Name: ______________________________

PID: ______________________________

1. Which of the following are correctly parenthesized according to the formal definintion of formulas? (Answer "Yes" or "No".)

Y (a) p_3
N (b) (p_3)
Y (c) $\neg p_3$
N (d) $(\neg p_3)$
N (e) $\neg p_3 \to p_4$
Y (f) $(\neg p_3 \to p_4)$
N (g) $(\neg p_3) \to p_4$
N (h) $((\neg p_3) \to p_4)$

2. Add parentheses to make these correct formulas (using the class's conventions on precedence of operations).

(a) $((\neg p_1 \vee p_2) \leftrightarrow (p_1 \wedge \neg p_2))$
(b) $((\neg p_2 \vee p_1) \to ((\neg p_1 \wedge \neg p_2) \to (p_6 \wedge \neg p_7)))$

3. Give truth assignments φ which show that:

(a) $(p_1 \vee p_2) \wedge (\neg p_1 \vee p_3)$ is satisfiable. $\varphi(p_1) = \varphi(p_2) = \varphi(p_3) = T$

(b) $(p_1 \vee p_2) \wedge (\neg p_1 \vee p_3)$ is not a tautology. $\varphi(p_1) = \varphi(p_2) = \varphi(p_3) = F$

4. Give truth tables which show that the following two formulas are tautologies.

(a) $(p_1 \to p_2) \leftrightarrow (\neg p_2 \to \neg p_1)$.

(b) $p_1 \to p_2 \to p_1$.

p_1	p_2	$p_1 \to p_2$	$p_1 \to (p_2 \to p_1)$	$\neg p_2 \to \neg p_1$	$(p_1 \to p_2) \leftrightarrow (\neg p_2 \to \neg p_1)$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Comments/concerns/questions/feedback to the instructor:

Therefore $\varphi(p_1 \to p_2) = \varphi(\neg p_2 \to \neg p_1)$

"Contrapositive"

$p_1 \to p_2 \models \neg p_2 \to \neg p_1$ $\quad$ $\neg p_2 \to \neg p_1 \models p_1 \to p_2$

Math 160A - Fall 2021 - Class Work - In Lecture, September 30
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

1. Indicate whether true or false. For those that are false, give truth assignments that show they are false. (Write T or F on the lines.)

F (a) $p \to q \vDash q \to p$. $\varphi(p) = F$ $\varphi(q) = T$

F (b) $p \to q \vDash \neg p \to \neg q$. $\varphi(p) = F$ $\varphi(q) = T$

F (c) $p \leftrightarrow q \vDash p \vee q$. $\varphi(p) = F$ $\varphi(q) = F$ ✓

T (d) $\neg(p \leftrightarrow q) \vDash p \vee q$.

2. Each formula in the left column is equivalent to a formula in the right column. Indicate which one by writing one of "1" through "4" on the lines.

2 a. $(p \to q) \to r$. 1. $p \wedge q \to r$.

1 b. $p \to (q \to r)$. 2. $(\neg p \vee q) \to r$. $\neg r \to \neg q \vDash\dashv q \to r$

4 c. $p \to q \vee r$. 3. $p \to q \wedge r$.

1 d. $p \to (\neg r \to \neg q)$. 4. $(p \wedge \neg q) \to r$.

For c, use method of truth tables
Or: Split into cases: $\varphi(p) = T$ and $\varphi(p) = F$.

3. Let Γ be the set of formulas $\{p_i \to p_{i+1} : i \geq 1\}$. I.e., $\Gamma = \{p_1 \to p_2, p_2 \to p_3, p_3 \to p_4, \ldots\}$. First, describe what truth assignments φ satisfy Γ.

#1) $\varphi(p_i) = F$, $\forall i$.

More generally
$\varphi_j(p_i) = T$ iff $i \geq j$ (φ is like φ_∞).

Then, for each of the following indicate whether true or false.

Yes (a) Γ is satisfiable.

Yes (b) $\Gamma \vDash p_2 \to p_3$.

No (c) $\Gamma \vDash p_3 \to p_2$.

Yes (d) $\Gamma \vDash \neg p_1 \vee p_2$.

No (e) $\Gamma \vDash \neg p_2 \vee p_1$.

No (f) $\Gamma \vDash p_3 \to p_1 \vee p_2$.

Yes (g) $\Gamma \vDash p_1 \vee p_2 \to p_3$.

Yes (h) $\Gamma \vDash p_1 \to p_2 \to p_1$. $\leftarrow$ A tautology

Comments/concerns/questions/feedback to the instructor: $\vDash p_1 \to p_2 \to p_1$

Math 160A - Fall 2021 - Class Work - In Lecture, October ~~4~~ 5

(Hand in for the instructor to upload to Gradescope)

Name:

PID:

1. Express the following formulas in CNF form and in DNF form:

(a) $p \leftrightarrow q$

(b) $p \to (q \to p)$. [Hint: It is a tautology.]

(c) $p \wedge q \to r$. [Hint: Only one truth assignment to p, q, r falsifies this formula.]

(a) $(p \vee \neg q) \wedge (\neg p \vee q)$ $\quad$ $(p \wedge q) \vee (\neg p \wedge \neg q)$

(b) ~~$p \models\!\dashv p \to (q \to p)$???~~

$p \vee \neg p \models\!\dashv p \to (q \to p)$

$p \vee \neg p$ is both a CNF and a DNF formula!

(c) $\neg p \vee \neg q \vee r$

- this formula is also both a CNF formula and a DNF formula.

-The straightforward construction of a DMF formula for (c) would give a DNF formula with 7 disjuncts, each a conjunction of 3 literals.

2. The NOR connective is dual to NAND, similarly to the way that $\vee$ is dual to $\wedge$. The symbol $\downarrow$ is used to denote NOR, and $\varphi(p \downarrow q)$ is defined to equal $\varphi(\neg(p \vee q))$.

- Show how to express $\neg p$ and $p \vee q$ using only the NOR connective $\downarrow$.
- Conclude that $\{\downarrow\}$ is adequate (truth-functionally complete).

$\neg A \models\!\dashv A \downarrow A$

$A \vee B \models\!\dashv (A \downarrow B) \downarrow (A \downarrow B)$

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, October 7
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

1. Show that $\{Maj^3\}$ is not adequate.

If $\varphi(p_i) = T$ for all i, then $\varphi(A) = T$ for any $\{Maj^3\}$-formula.

p_1	$Maj^3(p_1, p_1, p_1)$
T	T
F	F

Also: if $\varphi(p_i) = F$ for all i, then $\varphi(A) = F$ for any $\{Maj^3\}$-formula.

2. Show that $\{\neg, Maj^3\}$ is not adequate.

The constant $\perp$ is not equivalent to any $\{\neg, Maj^3\}$-formula

p_1	$Maj^3(p_1, p_1, p_1)$	$\neg Maj^3(p_1, p_1, p_1)$	$Maj^3(\neg p_1, p_1, p_1)$	$Maj^3(\neg p_1, \neg p_1, p_1)$	$Maj^3(\neg p_1, \neg p_1, \neg p_1)$
T	T	F	T	F	F
F	F	T	F	T	T

~~The only~~ Every $\{Maj^3, \neg\}$-formula on p_1, p_2 is equivalent to one of $p_1, p_2, \neg p_1$ or $\neg p_2$

3. Give a DNF formula tautologically equivalent to $Maj^3(p, q, r)$.

$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$.

$(p \wedge q) \vee (q \wedge r) \vee (p \wedge r)$.

For further discussion: Give a CNF formula tautologically equivalent to $Maj^3(p, q, r)$.

~~[illegible]~~

$(p \vee q) \wedge (q \vee r) \wedge (p \vee r)$

"at most ~~of~~ p, q, r is false"

"choose any two of p, q, r; at least one is true!"

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, October 12
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

1. Show $\vdash (A \to B) \to (A \to A)$ by giving an explicit PL-proof.
[Hint: The PL-proof has three lines, i.e. three formulas.]

$(A \to B \to A) \to (A \to B) \to (A \to A)$ PL 2

$A \to B \to A$ PL 1

$(A \to B) \to (A \to A)$

2. Prove that $A \to B, C \vdash A \to (C \to B)$. Did you need both the hypotheses? [Use the Deduction Theorem.] No

By the Deduction Theorem, it suffices to show:

$A \to B, C, A \vdash C \to B$

Suffices to show. $A \to B, C, A, C \vdash B$

This holds by Modus Ponens. $A \to B, A \vdash B$

so $A \to B, A, C \vdash B$

3. Prove that $A \to B \to C \vdash B \to (A \to C)$. twice!

By the Deduction Theorem, it suffices to show.

$A \to B \to C, B, A \vdash C$

This follows by 2 uses of Modus Ponens.

4. Prove that $A \to B \vdash (\neg A \to A) \to B$.

Suffices to show $\boxed{A \to B, \neg A \to A \vdash B}$ by the Deduction Theorem

Hypothetical Syllogism $A \to B, \neg A \to A \vdash \neg A \to B$ (Doesn't work yet!)

$\boxed{PL\ 4: \vdash (\neg A \to A) \to A}$

$\boxed{A \to B, \neg A \to A \vdash A}$ by Modus Ponens

$\boxed{A \to B, \neg A \to A \vdash B}$ by Modus Ponens again.

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, October 14
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

PL axioms:
PL1: $A \to B \to A$
PL2: $(A \to B \to C) \to (A \to B) \to (A \to C)$
PL3: $\neg A \to A \to B$
PL4: $(\neg A \to A) \to A$
• $A \vee B$ and $A \wedge B$ stand for $\neg A \to B$ and $\neg(A \to \neg B)$.

1. Prove that $\{\neg(A \to \neg\neg A)\}$ is inconsistent. (This is the same as $\{A \wedge \neg A\}$.)

THIS IS THE CORRECTED VERSION. SEE THE NEXT PAGE FOR THE IN-CLASS VERSION.

2. Prove that $\{\neg(A \to B), \neg A\}$ is inconsistent.

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, October 14
(Hand in for the instructor to upload to Gradescope)

PL axioms:
PL1: $A \to B \to A$
PL2: $(A \to B \to C) \to (A \to B) \to (A \to C)$
PL3: $\neg A \to A \to B$
PL4: $(\neg A \to A) \to A$
• $A \vee B$ and $A \wedge B$ stand for $\neg A \to B$ and $\neg(A \to \neg B)$.

Name:

PID:

$\neg(A \to \neg\neg A)$

1. Prove that $\{\neg(A \to \neg A)\}$ is inconsistent. (This is the same as $\{A \wedge \neg A\}$.)

~~Suffices to show~~ ~~$\vdash A \to \neg A$~~

Suffices to show: $\vdash A \to \neg\neg A$. $\boxed{1c}$ We proved this in class.

Since: $\emptyset \vdash (A \to \neg\neg A)$ iff $\{\neg(A \to \neg\neg A)\}$ is inconsistent (Proof by contradiction #1).

2. Prove that $\{\neg(A \to B), \neg A\}$ is inconsistent.

Suffices to show $\neg A \vdash A \to B$ (Proof by Contradiction #1)

Thus, suffices to show $\neg A, A \vdash B$ by Deduction Theorem.
→ This has already been shown.

(or suffices to show: $\vdash \neg A \to A \to B$, also by Deduction Theorem)
this is PL 3

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, October 19
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

PL axioms:
PL1: $A \to B \to A$
PL2: $(A \to B \to C) \to (A \to B) \to (A \to C)$
PL3: $\neg A \to A \to B$
PL4: $(\neg A \to A) \to A$
• $A \vee B$ and $A \wedge B$ stand for $\neg A \to B$ and $\neg(A \to \neg B)$.

1. Suppose $\Gamma \cup \{A\}$ and $\Gamma \cup \{\neg A\}$ are both inconsistent. Prove that Γ is inconsistent.

Therefore if Γ is consistent, then at least one of $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is consistent

Assume $\Gamma \cup \{A\}$ is inconsistent and $\Gamma \cup \{\neg A\}$ is consistent.

Thus $\Gamma \vdash \neg A$ and $\Gamma \vdash A$ by Proof by Contradiction Version 1 + 2

So Γ is inconsistent!

qed

Our Γ in ~~test~~ lecture

2. Suppose Γ_i, $i \geq 1$, are sets of formulas and that, for all i, $\Gamma_i \subseteq \Gamma_{i+1}$. Let $\Gamma = \bigcup_i \Gamma_i$. Also suppose Γ is ~~unsatisfiable~~ inconsistent. Prove that, for some i, Γ_i is ~~unsatisfiable~~ inconsistent. $\equiv$

So if each Γ_i is consistent, $\Gamma_i \subseteq \Gamma_{i+1}$, then $\bigcup \Gamma_i$ is consistent

Suppose Γ is inconsistent.

So $\Gamma \vdash \neg(p_1 \to p_1)$ (Fact: $\Gamma \vdash \neg(p_1 \to p_1)$ iff Γ is inconsistent)

So $\exists \Delta \subseteq \Gamma$, Δ is finite, $\Delta \vdash \neg(p_1 \to p_1)$

Let $\Delta = \{C_1, \ldots, C_k\}$, each C_j is in some Γ_{i_j}.

Let $I = \max_j (i_j)$. Then each $C_j \in \Gamma_I$

So $\Gamma_I \vdash \neg(p_1 \to p_1)$, i.e. Γ_I is inconsistent

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, October 26
(Hand in for the instructor to upload to Gradescope)

Name:

John likes his mother

PID:

Likes (John, Mother (John))

1. Use unary predicates $Dog(x)$, $Cat(x)$, $Person(x)$, the binary predicate $Likes(x, y)$, the constant symbols *John* and *Mary*, the unary function $Mother(x)$ and the equality sign $=$ to express the following English sentences in first-order logic.

(a) *John* is neither a cat nor a dog.

$\neg(Cat(John) \vee Dog(John))$

(b) All cats dislike all dogs.

$\underline{\forall x \forall y} \, (Cat(x) \wedge Dog(y) \to \neg Likes(x,y))$.

(c) *John* likes every person.

$\forall x \, (Person(x) \to Likes(John, x))$

(d) *John* likes anyone who likes all cats

anyone allow dogs & cats & umpeta s.

$\forall x (\underbrace{(x \text{ likes all cats})} \to Likes(John, x))$

$\forall x (\underbrace{(\forall y (Cat(y) \to Likes(x,y))} \to Likes(John, x))$

(e) Everyone is liked by their mother.

$\forall x \, (Likes(Mother(x), x))$

(f) Everyone is liked by their mother's mother.

$\forall x \, (Likes(Mother(Mother(x)), x)$

(g) *John* likes everyone who has the same mother (as John).

$\forall x \, (Mother(x) = Mother(John) \to Likes(John. x))$

(h) *Mary* likes each person that John likes.

$\forall x \, (Likes(John, x) \to Likes(Mary, x))$

$\forall x (Person(x) \wedge Likes(John, x) \to Likes(Mary, x))$

(i) *John* likes someone only if they like themselves.

$\forall x \, (Likes(John, x) \to Likes(x,x))$

(j) *John* likes precisely those who dislike themselves.

$\forall x \, (Likes(John, x) \leftrightarrow \neg Likes(x,x))$ $\leftarrow$ $\underline{\text{Contradictory}}$

Take $x := John$

$Likes(John, John)$

$\leftrightarrow \neg Likes(John, John)$

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, October 28
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

1. Let L_{PA} be the language $\{0, S, +, \cdot\}$ (zero, successor, addition, multiplication). Give first-order formulas that express the following properties over the non-negative integers.

1. Express $x \leq y$ as an L_{PA}-formula.

$\exists z(x+z=y)$

2. Express $x|y$, "x is a divisor of y (y is a multiple of x)," as an L_{PA}-formula.

bound.

$\exists z(x \cdot z = y)$

free

3. Express $x = lcm(y, z)$, "x is the least common multiple of y and z", as an $L_{\mathrm{PA}} \cup \{\leq, |\}$-formula.

$y|x \wedge z|x \wedge \forall w(y|w \wedge z|w \rightarrow x \leq w).$

4. Express $Prime(x)$, "x is a prime number", as an $L_{\mathrm{PA}} \cup \{\leq, |\}$-formula.

$\forall y(y|x \rightarrow y=S(0) \vee y=x) \wedge x \neq S(0)$

Prime(0) is false
i.e $y=2$,
$2|0$ because $2\cdot 0=0$

5. Express "x is a prime number and y is a power of x", as an $L_{\mathrm{PA}} \cup \{\leq, |, Prime\}$-formula.

$Prime(x) \wedge \forall z(z|y \rightarrow x|z \vee z=S(0))$

(The property "y is a power of x" can also be expressed by an L_{PA}-formula (without the condition that x is prime), but it is much more difficult to do)

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, November 2
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

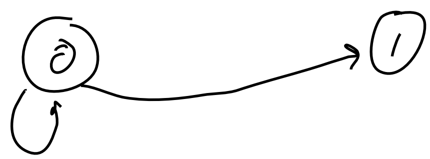
We define two structures $\mathfrak{A}$ and $\mathfrak{B}$. They both use the language $L = \{E\}$ where E is a binary predicate symbol. We think of this as being the language for directed graphs where $E(x, y)$ means there is a directed edge from x to y.

I. The structure $\mathfrak{A}$ has $|\mathfrak{A}| = \{0, 1\}$, and $E^{\mathfrak{A}} = \{\langle 0, 1\rangle, \langle 0, 0\rangle\}$.

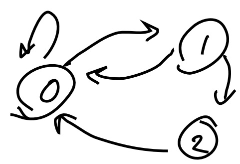
- Draw a picture of A.
- Is the sentence $\forall x\, \exists y\, E(x, y)$ true in (satisfied by) $\mathfrak{A}$? I.e., Does $\mathfrak{A} \models \forall x\, \exists y\, E(x, y)$ hold? No $\sigma(x) = 1$

  $\mathfrak{A} \not\models \exists y\, E(x,y)[\sigma]$
- Does $\mathfrak{A} \models \forall x\, \exists y\, E(y, x)$ hold?

  Yes



II. The structure $\mathfrak{B}$ has $|\mathfrak{B}| = \{0, 1, 2\}$, and $E^{\mathfrak{A}} = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle, \langle 1, 2\rangle, \langle 2, 0\rangle\}$.
Draw a picture of $\mathfrak{B}$.
Suppose $\sigma(x_1) = 0$, $\sigma(x_2) = 1$, $\sigma(x_3) = 2$ and $\sigma(x_4) = 0$.
Which of the following are correct statements?



- $\mathfrak{B} \models x_1 = x_4[\sigma]$. Yes $\sigma(x_1) = 0$ $\sigma(x_4) = 0$
- $\mathfrak{B} \models \forall x_2\, E(x_1, x_1)[\sigma]$. Yes $\sigma(x_1) = 0$. $\tau(x_2)$ doesn't matter
- $\mathfrak{B} \models \forall x_1\, E(x_1, x_1)[\sigma]$. No $\tau(x_1) = 2$ or $\tau(x_1) = 1$
- $\mathfrak{B} \models \forall x_1\, (E(x_2, x_1) \to E(x_1, x_1))[\sigma]$. $\sigma(x_2) = 1$

No, try $\tau(x_1) = 2$

$\mathfrak{B} \not\models E(x_2, x_1) \to E(x_1, x_1)[\tau]$

$\mathfrak{B} \not\models \forall x_1\, E(x_1, x_1)$

$\iff \mathfrak{B} \models \exists x_1\, \neg E(x_1, x_1)$.

III. Give examples of sentences that are satisfied by $\mathfrak{A}$ but not by $\mathfrak{B}$.

$\forall x_1((\exists x_2 E(x_2, x_1)) \to E(x_2, x_2)))$ – ?

$\forall x_1 \forall x_2 \forall x_3 (x_1 = x_2 \vee x_2 = x_3 \vee x_1 = x_3)$. ≤ 2 elements

$\neg \forall x_1 \exists x_2\, E(x_1, x_2)$ — Some vertex has no outgoing edge.

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, November 4
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

~~We define two structures $\mathfrak{A}$ and $\mathfrak{B}$. They both use the language $L = \{E\}$ where E is a binary predicate symbol. We think of this as being the language for directed graphs where $E(x, y)$ means there is a directed edge from x to y.~~

I. Work with the language $L = \{0, +, \cdot\}$. Consider the following four L-structures.

- The reals $\mathcal{R} = (\mathbb{R}, 0, +, \cdot)$.
  I.e., $|\mathcal{R}| = \mathbb{R}$, and $0^{\mathcal{R}} = 0$ and $+^{\mathcal{R}} = \{\langle a, b, c\rangle : a, b, c \in \mathbb{R} \text{ and } a + b = c\}$, and $\cdot^{\mathcal{R}} = \{\langle a, b, c\rangle : a, b, c \in \mathbb{R} \text{ and } a \cdot b = c\}$.
- The rationals $\mathcal{Q} = (\mathbb{Q}, 0, +, \cdot)$.
- The integers $\mathcal{Z} = (\mathbb{Z}, 0, +, \cdot)$.
- The non-negative integers $\mathcal{N} = (\mathbb{N}, 0, +, \cdot)$.

Give sentences that distinguish these four structures. Namely, for each structure find a sentence which is satisfied by just that structure and not by any of the other three.

A: $\exists x(\underbrace{x\cdot x = x \wedge x \neq 0}_{\text{forcing } x=1} \wedge \underbrace{\forall y(y + x \neq 0)}_{\text{False with } y=-1})$ - ~~True~~ False for R, Q, Z not for N.

"There is an element -1"

$x \cdot x = x \wedge x \neq 0$ - defines 1 in all 4 structures

$x \cdot x = x \wedge x + x = x$ - defines 0 in all 4 structures.

$\forall x \exists y (x \cdot y = 1)$ False in all 4 $(x = 0)$

B: $\forall x (x \neq 0 \rightarrow \exists y (x \cdot y = 1))$ True in R, Q, not in Z, N.

$(\neg A \wedge \neg B)$ - True for Z not R, Q, N

$\forall x \exists p \exists q (x \cdot p = q)$ - True for all 4. Take $p = 1$, $q = x$.

$\forall x \exists y (x = y \cdot y)$. "Everything has a square root"

Comments/concerns/questions/feedback to the instructor:

$\forall x \exists y (x = y \cdot y \cdot y)$. "Everything has a cube root" True R only.

Math 160A - Fall 2021 - Class Work - In Lecture, November 9
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

Adobe Acrobat crashed at the end of class, and handwritten annotations could not be saved. See the podcast for the annotations and discussion. The version below includes corrections to the typos discused in the lecture.

1. Let A be the formula $\exists x \exists y \exists z \exists u(x \cdot x + y \cdot y + z \cdot z + u \cdot u = v)$. This says v can be written as the sum of four squares. (A theorem of Lagrange states that every nonnegative integer can be written as the sum of four squares.)

(a) Give examples of terms that are and are not substitutable in A for v.

(b) Give a formula that expresses that $x + y$ is the sum of four squares.

2. Give an example of a formula B such that $B(y/x)(x/y)$ is not equal to B. Can you do this even if in both cases substitutability holds? Can you do it even if there are no "extra" occurrences of y?

3. (Parallel substitution versus sequential substitution.) Let C be $x_1 = x_2$. What is $C(x_2, x_3/x_1, x_2)$? What is $C(x_2/x_1)(x_3/x_2)$? Are they the same?

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, November 16
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

A *prenex formula* is a formula with all of its quantifiers at the beginning of the formula, namely $Qx_{i_1}Qx_{i_2}\cdots Qx_{i_k}B$ where B contains no quantifiers.

The tautological equivalences to the right allow expressing any first-order formula as a logically equivalent prenex formula. (Any use of $\leftrightarrow$ needs to be reexpressed in terms of $\wedge$ and $\rightarrow$.)

$$\begin{aligned}
\forall x\, A &\vDash\Dashv \neg\exists x\,\neg A\\
\exists x\, A &\vDash\Dashv \neg\forall x\,\neg A\\
\neg\forall x\, A &\vDash\Dashv \exists x\,\neg A\\
\neg\exists x\, A &\vDash\Dashv \forall x\,\neg A\\
C\wedge\exists x\, A &\vDash\Dashv \exists x\,(C\wedge A)\\
C\wedge\forall x\, A &\vDash\Dashv \forall x\,(C\wedge A)\\
C\vee\forall x\, A &\vDash\Dashv \forall x\,(C\vee A)\\
C\vee\exists x\, A &\vDash\Dashv \exists x\,(C\vee A)\\
C\rightarrow\forall x\, A &\vDash\Dashv \forall x\,(C\rightarrow A)\\
C\rightarrow\exists x\, A &\vDash\Dashv \exists x\,(C\rightarrow A)\\
(\forall x\, A)\rightarrow C &\vDash\Dashv \exists x\,(A\rightarrow C)\\
(\exists x\, A)\rightarrow C &\vDash\Dashv \forall x\,(A\rightarrow C)
\end{aligned}$$

1. Convert the following formulas to logically equivalent prenex formulas.

(a) ("Every dog knows a cat that likes John")

$$\forall x\,(Dog(x)\rightarrow\exists y(Cat(y)\wedge Knows(x,y)\wedge Likes(y,John))).$$

$\forall x\,\exists y\,(Dog(x)\rightarrow Cat(y)\wedge Knows(x,y)\wedge Likes(y, John)))$.

(b) ("Anyone who knows a cat that likes John is a dog")

$$\forall x\,(\exists y(Cat(y)\wedge Knows(x,y)\wedge Likes(y,John))\rightarrow Dog(x)).$$

$\forall x\,\forall y\,((Cat(y)\wedge Knows(x,y)\wedge Likes(y, John))\rightarrow Dog(x))$

(c) ("x is prime") $\quad x\neq S(0)\wedge\forall y(\exists z(y\cdot z=x)\rightarrow y=S(0)\vee y=x)$.

$\forall y\,\forall z\,(x\neq S(0)\wedge((y\cdot z=x)\rightarrow y=S(0)\vee y=x))$

(d) $\exists x Q(x,y)\rightarrow\exists z\,\forall y\,Q(y,z)$

$\exists x\, Q(x,y)\rightarrow\exists z\,\forall y\,Q(y,z)$

y - both free + bound !!

$\forall x\,\exists z\,(Q(x,y)\rightarrow\forall y\,Q(y,z))$

$\forall x\,\exists z\,(Q(x,y)\rightarrow\forall w\,Q(w,z))$ ← Alphabetic variant.

Comments/concerns/questions/feedback to the instructor:

$\forall x\,\exists z\,\forall w\,(Q(x,y)\rightarrow Q(w,z))$ ← Prenex!!

$\exists z\,\forall w\,\forall x\,(Q(x,y)\rightarrow Q(w,z))$

$\exists z\,\forall x\,\forall w\,(Q(x,y)\rightarrow Q(w,z))$

Math 160A - Fall 2021 - Class Work - In Lecture, November 23
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

1. Show that $P(x) \vdash Q(z) \to P(x)$.

2. Show $\forall x\, P(x) \vdash \forall y\, P(y)$.

$\vdash \forall x P(x) \to P(y)$ Axiom

$\vdash \forall x P(x) \to \forall y P(y)$ Generalization

$\vdash \forall x P(x)$ Hypothesis

$\vdash \forall y P(y)$ M.P.

3. Show that $\vdash x = y \to y = z \to z = x$.

$\vdash x = y \to y = z \to x = z$ Transitivity axiom.

$\vdash x = z \to z = x$ Symmetry axiom.

$\vdash x = y \to y = z \to z = x$ TAUT

$\dfrac{A \to B}{\neg B \to \neg A}$ Contr

$\dfrac{A \to \neg B}{B \to \neg A}$ TAUT

4. Let $A = A(x)$. Show that $\vdash A(t) \to \exists x\, A(x)$.

I.e. Show $\vdash A(t) \to \neg \forall x \neg A(x)$.

Suggest: Use $\vdash \forall x \neg A(x) \to \neg A(t)$ - axiom.

By Contraposition. $\vdash \neg\neg A(t) \to \neg \forall x \neg A(x)$

Therefore $\vdash A(t) \to \neg \forall x \neg A(x)$.

Comments/concerns/questions/feedback to the instructor:

Math 160A - Fall 2021 - Class Work - In Lecture, November 30
(Hand in for the instructor to upload to Gradescope)

Name:

PID:

1. For each the following: Either (a) prove it has an FO proof (not using the Completeness Theorem), or (b) Prove it does not have FO-proof by giving a structure which falsifies it (that is, by using the Soundness Theorem).

- $\forall x\,(P(x) \to Q(x)) \to \forall x\,P(x) \to \forall x\,Q(x)$.
- $\exists x\,(P(x) \to Q(x)) \to \exists x\,P(x) \to \exists x\,Q(x)$.
- $\forall x\,(P(x) \to Q(x)) \to \exists x\,P(x) \to \exists x\,Q(x)$.
- $\exists x\,(P(x) \to Q(x)) \to \forall x\,P(x) \to \forall x\,Q(x)$.

Logically valid
Not logically valid

Provable. $\boxed{\text{Suffices to show } \forall x\,(P(x)\to Q(x)),\ \forall x P(x) \vdash \forall x\, Q(x)}$ by Deduction Thm twice.

Informally: Try to ~~prove~~ prove $Q(c)$
Know: $P(c)$ and $P(c)\to Q(c)$. So $Q(c)$. !

Let $\Gamma = \forall x\,(P(x)\to Q(x)),\ \forall x\, P(x)$

So $\Gamma \vdash \forall x\,(P(x)\to Q(x))$
So $\Gamma \vdash P(x)\to Q(x)$

$\forall x\,(P(x)\to Q(x)) \to (P(x)\to Q(x))$
is a UI axiom

Similarly, $\Gamma \vdash P(x)$
By M.P. $\Gamma \vdash Q(x)$
By Gen $\Gamma \vdash \forall x\,(Q(x))$

$$\frac{A}{\forall x A} \qquad \frac{C \to A}{C \to \forall x A}$$

Second one not logically valid. By Soundness it does not have a proof.

$|\mathfrak{A}| = \{0,1\}$

$P^{\mathfrak{A}} = \{\langle 0 \rangle\}$

$Q^{\mathfrak{A}} = \emptyset$.

Comments/concerns/questions/feedback to the instructor:

FO axioms and inferences:

PL1: $A \to B \to A$
PL2: $(A \to B \to C) \to (A \to B) \to (A \to C)$
PL3: $\neg A \to A \to B$
PL4: $(\neg A \to A) \to A$
EQ1: $x = x$
EQ2: $x = y \to y = x$
EQ3: $x = y \to y = z \to x = z$
EQ$_f$: $y_1 = z_1 \to \cdots \to y_k = z_k \to f(y_1, \ldots, y_k) = f(z_1, \ldots, z_k)$
EQ$_P$: $y_1 = z_1 \to \cdots \to y_k = z_k \to P(y_1, \ldots, y_k) \to P(z_1, \ldots, z_k)$
UI: $\forall x\, A(x) \to A(t)$.
MP: $A \to B,\ A \ /\ B$.
Gen: $C \to A \ /\ C \to \forall x\, A$ (x not free in C)

- $A \vee B$, $A \wedge B$, and $\exists x\, A$ stand for $\neg A \to B$, $\neg(A \to \neg B)$ and $\neg \forall x \neg A$.