Math 160A - Fall 2021 - Homework 7
Due Wednesday, November 17, 11:00am — DUE IN THE MORNING!
(Hand in by uploading to Gradescope)

1. Work in the language with a unary predicate $P(x)$ and (as usual) the equality sign $=$. For $k \geq 0$, let $AtMost_k$, $AtLeast_k$ and $Exactly_k$ be the assertions that there are at most k, at least k, and exactly k (respectively) many objects x satisfying the property $P(x)$.

(a) Give general constructions, for arbitrary fixed k, of first-order sentences that express the assertions $AtMost_k$, $AtLeast_k$ and $Exactly_k$.
(b) Analyze the sizes of the sentences you constructed for $AtMost_k$, $AtLeast_k$ and $Exactly_k$. Do they use $O(k^2)$ many symbols? Do they use $O(k)$ many symbols?[1]

(The most straightforward constructions use $O(k^2)$ many symbols, but it is not too hard to express them using only $O(k)$ many symbols. For size $O(k)$, the suggestion is to start with $AtMost_k$.)

2. Suppose x is not free in a formula A. Prove that $A \vDash\dashv \forall x\, A$ and that $A \vDash\dashv \exists x\, A$. [Part of this was already done in the previous homework assignment. There is an easy way to do the second part from the first part, by replacing subformulas with equivalent formulas using the fact that, loosely speaking, "$\neg\forall\neg$" is logically equivalent to "$\exists$", see Theorems IV.33 and IV.48 (was number IV.47 in the earlier PDF draft).]

3. Let A and B be arbitrary formulas. Prove that $\forall x\,(A \land B)$ is logically equivalent to $\forall x\, A \land \forall x\, B$. Then prove that $\exists x\,(A \lor B) \vDash\dashv \exists x\, A \lor \exists x\, B$. (Again, the second part can be proved from the first part, now using the equivalence of $\neg\forall\neg$ and $\exists$, and De Morgan's laws.)

4. Suppose x is not free in C. Prove that

(a) $\forall x(A \land C) \vDash\dashv \forall x A \land C$. (Hint: this one follows readily from problems 2 and 3.)
(b) $\exists x(A \land C) \vDash\dashv \exists x A \land C$. (Hint: This needs a new proof. Split into cases depending on whether or not $\mathfrak{A} \vDash C[\sigma]$ holds. **Extra Hint:** Example IV.38 of the PDF text shows a similar proof.)

5. Again suppose x is not free in C. Prove that

(a) $\forall x(A \lor C) \vDash\dashv \forall x A \lor C$
(b) $\exists x(A \lor C) \vDash\dashv \exists x A \lor C$

(Hint: use the equivalence of $\neg\exists\neg$ and $\forall$, De Morgan's laws, and problem 4.)

[1] $O(k)$ and $O(k^2)$ are "big-Oh notation" $O(k)$ means having size bounded by $c \cdot k$ for some constant c. $O(k^2)$ means having size bounded by $c \cdot k^2$ for some constant c.