Math 160A - Fall 2021 - Homework 6 - Due Wednesday, November 10, 10:00pm

(Hand in by uploading to Gradescope)

1. Use unary predicates $\mathit{Jazz}(x)$ and $\mathit{Kpop}(x)$, the binary predicates $\mathit{Likes}(x, y)$ and $\mathit{Knows}(x, y)$, the constant symbols *Joan* and *John*, and the equality sign = to express the following English sentences in first-order logic. $\mathit{Jazz}(x)$ means "x is a jazz musician", $\mathit{Kpop}(x)$ means "x is a K-pop musician", $\mathit{Likes}(x, y)$ means "x likes y", and $\mathit{Knows}(x, y)$ means "x knows y". Variables range over the universe of people.

Express the following as first-order formulas:

(a) Jazz musicians do not like K-pop musicians.
(b) Joan knows a jazz musician who likes every K-pop musician.
(c) John likes everyone he knows.
(d) There is no one that John and Joan both like.
(e) Everyone that John likes knows a jazz musican.

2. Use the same language as above. For each sentence (a)-(d), either: (i) if it is a syntactically correct formula, give a translation into English, or (ii) state that it is not a syntactic first-order formula.

(a) $\forall x\, \exists y\, \mathit{Likes}(x, \mathit{Jazz}(y))$.
(b) $\forall x\, (\mathit{Likes}(\mathit{John}, x) \leftrightarrow \mathit{Likes}(\mathit{Joan}, x))$.
(c) $\forall x\, \forall x\, \mathit{Likes}(\mathit{Jazz}(x), \mathit{Jazz}(y))$.
(d) $\forall x\, (\mathit{Likes}(\mathit{John}, x) \rightarrow \exists y\, (\mathit{Jazz}(y) \wedge \mathit{Likes}(x, y)))$.

3. Show that none of (a), (b) or (c) is logically implied by the other two. Do this by giving structures $\mathfrak{A}$ which satisfy only two of the three sentences.

(a) $\forall x\, \forall y\, \forall z\, (x \neq y \rightarrow (P(x, y) \leftrightarrow \neg P(y, x))$.
(b) $\exists x\, \forall y\, (\neg P(x, y))$.
(c) $\forall x\, \forall y\, \forall z\, (P(x, y) \rightarrow P(y, z) \rightarrow P(x, z))$.

4. Show that $\forall y\, E(x, y) \wedge \neg E(z, x)$ is satisfiable by explicitly describing a pair $(\mathfrak{A}, \sigma)$ that satisfies it.

5. Which of the following are tautologies?

(a) $P(x, x) \rightarrow \exists y\, (P(y, y) \vee P(x, x))$.
(b) $\forall x\, \forall y\, (P(x, y) \rightarrow P(x, y) \rightarrow P(x, y))$.
(c) $P(x, y) \rightarrow P(x, y) \rightarrow P(x, y)$.

6.

(a) Give an example of a formula A so that $A \not\models \forall x A$. Give an example of pair $(\mathfrak{A}, \sigma)$ which demonstrates that $A \not\models \forall x A$.
(b) Suppose that x does not appear free in the formula A. Prove that $A \models \forall x A$.