Math 160A - Fall 2021 - Homework 4 - Due Wednesday, October 27, 10:00pm
(Hand in by uploading to Gradescope)

1. Suppose that $\Gamma \models p_i$ or $\Gamma \models \neg p_i$, for every i. Prove that, for every formula A, $\Gamma \models A$ or $\Gamma \models \neg A$. (This property of Γ is similar to being complete; however, instead of having one of A or $\neg A$ a member of Γ, we have one of A or $\neg A$ tautologically implied by Γ.)

2. Use the Compactness Theorem for propositional logic to prove that a graph is 3-colorable if and only if every finite subgraph is 3-colorable. ("3-colorable" means there is an assignment of three colors to the vertices of the graph so that no edge connects vertices assigned the same color.) For this, fix a graph G. Use propositional variables r_i, g_i, b_i whose intended meanings are that "Vertex i is red", "Vertex i is green", and "Vertex i is blue", respectively. Let Γ be a set of formulas using these variables that expresses the conditions that (a) each vertex has a color assigned to it, and (b) if two vectices i and j are joined by an edge in G, then they are not assigned the same color. The set Γ should be satisfiable if and only if G is 3-colorable. Then apply the Compactness Theorem.

This is mostly a conceptual problem. Feel free to discuss this on piazza and discord. What to hand-in to be graded: Describe what formulas are in the set Γ in terms of the vertices and edges of G.