Math 160A - Winter 2016 - Homework #6
Instructor: Sam Buss - UC San Diego

Due Wednesday, February 24, 2016, 7:00pm.

This is a multipart homework, intended to give you some practice with working with first order logic and their meaning in structures.

The last three pages of this are three distinct homeworks, Homework 6A, Homework 6B and Homework 6C. You may turn these in in class on Monday or Wednesday, or online on the GradeScope program. If you turn them in online with the GradeScope program, I will endeavor to grade and return it to you within 24 hours.

Background for Homework 6A & 6B: Directed graphs. Directed graphs are specified with a binary predicate $E(x, y)$, meaning that there is a directed edge from vertex x to vertex y. We will also use predicates $R(x)$, $G(x)$ and $B(x)$ to mean that vertex x is colored red, green, or blue, respectively.

To specify a structure $\mathfrak{A}$, give explicitly the universe $|\mathfrak{A}|$, say by listing its elements if it is finite. To answer homework assignments, you should explicitly list the edges in $E^{\mathfrak{A}}$, if possible, or finding some other way to describe it as a set of ordered pairs. (For an example, see page 82 in the textbook.) You may may find it helpful to also describe the graph by drawing it with vertices and arrows. (This also as an example on page 82 in the textbook.)

If the language contains R, G, B, then the description of the structure must also contain the descriptions of $R^{\mathfrak{A}}$, $G^{\mathfrak{A}}$, $B^{\mathfrak{A}}$ as sets, preferably by listing their members explicitly in the finite setting.

Background for Homework 6C: Partial and linear orders and lattices. For this problem, work with the language with equality ($=$) and a binary relation $\leq$. You should have seen all this terminology before, but if not, check the Wikipedia pages for "partially ordered set" and "Lattice (order)" and "Equivalence relation".

Homework 6A — Math 160A — Winter 2016

NAME: ____________________

Problem 6A. Give first-order formulas expressing the following properties. For b., the *neighborhood of* x consists of x plus all vertices y adjacent to x (i.e. all vertices y such that there is at least one edge (either direction is OK) between x and y). For the last three, use the language with unary predicates R, G, B.

a. x is an isolated vertex (with no incident edges *except* possibly a self-loop).

b. The neighborhood of x is the entire graph.

c. There is a sink vertex, but no source vertex.

d. Any two distinct vertices are joined by a directed path of length two. (The path could go in either direction.)

e. x is a vertex with at least two colors.

f. Every vertex has at least one color.

g. The coloring of the graph is a proper coloring (no edge connects two vertices that have a common color).

Homework 6B — Math 160A — Winter 2016

NAME: ____________________

Problem 6B. Explicitly give structures showing the following logical non-implications. For full credit, give structures with as few nodes as possible.

a. $\not\models \forall x \, \forall y \, (x \neq y \rightarrow E(x,y) \vee E(y,x))$.

b. $\forall x \, \exists y E(y,x) \not\models \exists x \, \forall y (x \neq y \rightarrow \neg E(x,y))$.

c. $\forall x [\exists y E(x,y) \wedge \forall y \forall z (E(x,y) \wedge E(z,y) \rightarrow x = z)] \not\models \forall y \exists x E(x,y)$.

d. For c., your answer is an infinite structure. Why is this?

Homework 6C — Math 160A — Winter 2016

NAME: ____________________

Problem 6C. Work in the language with binary symbols $\leq$ and $=$. Give first-order formulas expressing the following properties. You may find it useful to give your answers names, and reuse these formulas by name in later answers. (E.g., the answer to f. can incorporate the answers to a., b., and d., and not need to write out the full formula.)

a. $\leq$ is reflexive.

b. $\leq$ is antisymmetric.

c. $\leq$ is symmetric. (Don't be put off by the fact that $\leq$ is rarely used as a name for a symmetric relation.)

d. $\leq$ is transitive.

e. $\leq$ is a partial order.

f. $\leq$ is a linear order (i.e., a total order).

g. $\leq$ is an equivalence relation.

h. x and y have a least upper bound.

i. $\leq$ is a dense linear order: "dense" means that for every distinct x and y with $x \leq y$, there is a z distinct from x and y such that $x \leq z \leq y$.