Name: ______________________________ ID#:____________________

Math 15A - Discrete Mathematics - Spring 1999

Quiz #5 — May 12 - Answer Key

You may NOT use the textbook, notes or other references for this test. As usual, proofs are graded on form as well as content. It might be useful to note that (1) if n is even, then $\lfloor n/2 \rfloor = \lceil n/2 \rceil = n/2$, and that (2) if n is odd, then $\lfloor n/2 \rfloor = n/2 - 1/2$ and $\lceil n/2 \rceil = n/2 + 1/2$. (If you use these facts, you do not need to give a proofs of them.)

1. (10 pts) Define the infinite sequence of values e_n for $n \geq 1$ as follows: let $e_1 = 2$ and, for $n \geq 2$, let $e_n = e_{\lfloor n/2 \rfloor} \cdot e_{\lceil n/2 \rceil}$.

   Give the values of e_2, of e_3 and of e_4.

   ANSWER: $e_2 = 4$, $e_3 = 8$, $e_4 = 16$.

2. (50pt) Prove the following theorem:

   Theorem: Let e_n be defined as above. Then $e_n = 2^n$ for all integers $n \geq 1$.

   Proof: We will prove that $e_n = 2^n$ for all $n \geq 1$ by strong induction on n.

   Base case: $n = 1$. $e_1 = 2 = 2^1$ is immediate.

   Induction step: Let $n > 1$. Assume $e_k = 2^k$ for all positive $k < n$. The argument splits into two cases:

   Case 1: Suppose n is even. Then

   $$e_n \;=\; e_{n/2} \cdot e_{n/2} \;=\; 2^{n/2} 2^{n/2} \;=\; 2^n.$$

   The second equality used the induction hypothesis with $k = n/2$.

   Case 2: Suppose n is odd. Then

   $$e_n \;=\; e_{n/2-1/2} \cdot e_{n/2+1/2} \;=\; 2^{n/2-1/2} 2^{n/2+1/2} \;=\; 2^n.$$

   The second equality used the induction hypothesis twice, with $k = n/2 - 1/2$ and $k = n/2 + 1/2$.

   Q.E.D.

   Remark: We used in both cases the observation about $\lfloor n \rfloor$ and $\lceil n \rceil$ made above, plus the fact that $2^a 2^b = 2^{a+b}$.