Name: ______________________________ ID#:__________________

Math 15A - Discrete Mathematics - Spring 1999

Answers for Midterm Review Problems — May 3

On the midterm, you may use a *single* sheet of notes (double-sided, $8\frac{1}{2} \times 11$ paper). You may not use the textbook or a calculator. This sheet may not be shared with other students: you must have your own separate sheet of notes.

The midterm will cover all the course material to-date, through section 4.3.

1. (a) Consider the sequence $1, -2, 4, -8, 16, -32, \ldots$. Letting $a_0 = 1$, $a_1 = -2$, etc., give a general formula for a_i.

   ANSWER: $a_i = (-2)^i$.

   (b) Give a formula for $\sum_{i=0}^{n} a_i$.

   ANSWER: $\dfrac{(-2)^{n+1} - 1}{-3}$.

2. (a) Express the integer 85 in binary (base 2) notation.

   ANSWER: 1010101.

   (b) Express the negative integer -85 in binary notation (using a 8-bit two's complement representation).

   ANSWER: 10101011

3. Convert the propositional formula

   $$p \to (q \to (r \to (s \to t)))$$

   into an equivalent formula using only the connectives $\wedge$, $\vee$ and $\sim$.

   ANSWER: $\sim(p \wedge q \wedge r \wedge s) \vee t$.

   ANOTHER ANSWER: $\sim p \vee \sim q \vee \sim r \vee \sim s \vee t$

4. Express the formula $(p \leftrightarrow q) \leftrightarrow r$ as a Boolean circuit using AND, OR and NOT gates.

   HINT: use the fact that $(p \leftrightarrow q)$ is the same as (p AND q) OR (NOT p AND NOT q). The output of that circuit computing $(p \leftrightarrow q)$ is input along with r into another copy of the same circuit. The resulting circuit has 10 gates.

5. Let $L(x, y)$ be the predicate "x loves y", and let $H(x)$ be the predicate "x is handsome". Let quantified variables in $(\forall x)$ and $(\exists y)$ range over all people. Express the following formally using quantifiers, Boolean connectives and the two predicates:

   (a) Someone loves everyone.

   ANSWER: $\exists x \forall y L(x, y)$.

(b) Everyone is loved by someone.

ANSWER: $\forall x \exists y L(y, x)$.

(c) Some people love no one.

ANSWER: $\exists x \forall y \sim L(x, y)$.

(d) Any handsome person is loved by everyone.

ANSWER: $\forall x(H(x) \rightarrow \forall y(L(y, x)))$.

(e) There is a handsome person who is loved by no one.

ANSWER: $\exists x(H(x) \wedge \forall y(\sim L(y, x)))$.

(f) For a person to be loved by everyone it is necessary that they be handsome.

ANSWER: $\forall x((\forall y(L(y, x))) \rightarrow H(x))$.

6. Which of the following are tautologies?

(a) $(\sim(p \wedge \sim q)) \leftrightarrow (\sim p \vee q)$

(b) $p \rightarrow ((p \rightarrow q) \rightarrow p)$

(c) $(p \leftrightarrow (q \leftrightarrow r)) \leftrightarrow ((p \leftrightarrow q) \leftrightarrow r)$.

ANSWERS: ALL THREE ARE TAUTOLOGIES.

7. Express the following assertions (semi-)formally. You may use quantifers such as "$\forall$ rational x", "$\exists$ positive integer y", Boolean connections, and predicates such as "$x < y$", "$x \leq y$", "$x = y$" "$x \neq y$", and functions addition, subtraction, multiplication, division and square roots.

(a) There is a smallest positive integer.

ANSWER: $\exists$ a positive integer x s.t. $\forall$ positive integers y, $x \leq y$.

(b) There is no smallest positive rational number.

ANSWER: $\forall$ rationals x, $\exists$ a rational y s.t. $y < x$.

(c) For any irrational $x < y$, there is a rational z between x and y.

ANSWER: $\forall$ irrationals $x < y$, $\exists$ a rational z s.t. $x < z < y$.

(d) For an integer n to be prime, it is sufficient that n has at most one divisor which is greater than or equal to $\sqrt{n}$.

ANSWER: This is problem is harder than I intended it to be. To answer it the way I asked, you will need to expand out the definition of "prime" and "divides". The medium length answer is:
$\forall$ integers n, if ($\forall$ integers x, y , if $\sqrt{n} < x, y$ and $x|n$ and $y|n$ then $x = y$), then n is prime.

8. Which of the assertions (a)-(d) of the previous problem are true?

ALL OF THEM ARE TRUE.

9. Prove or disprove the following assertion:

   Assertion: Let n be an integer. Then $n^3 \bmod 8$ is equal to $n \bmod 8$.

   THIS IS TRUE. PROVE IT BY CASES! IN THE MOST STRAIGHTFORWARD PROOF THERE ARE 4 CASES DEPENDING ON THE VALUE OF $n \bmod 8$.

   Proof. Let n be an integer. Since n is odd, $n \bmod 8$ is odd and hence equal to 1, 3, 5, or 7. (If $n \bmod 8$ were even, then $n = 8k + p$ for some integers k, p with p even, and then n is the sum of two even integers and thus even.)

   Case 1 Assume $n \bmod 8$ equals 1. Then $n = 8p + 1$ for some integer p. And $n^3 = 8(64n^3 + 3 \cdot 8n^2 + 3n) + 1$, so $n^3 \bmod 8$ is also equal to 1.

   Case 2 Assume $n \bmod 8$ equals 3. Then $n = 8p + 3$ for some integer p. And $n^3 = 8(64n^3 + 3 \cdot 3 \cdot 8n^2 + 3 \cdot 3^2 n + 3) + 3$, so $n^3 \bmod 8$ is also equal to 3.

   Case 3 Assume $n \bmod 8$ equals 5. *This is similar to the previous cases and I omit the argument from this answer sheet.*

   Case 4 Assume $n \bmod 8$ equals 7. *This is similar to the previous cases and I omit the argument from this answer sheet.*

10. Prove or disprove the following assertion: (You may assume without proof that $\sqrt{2}$ is irrational if you need to. Any other needed irrational number should be proved to be irrational.)

    Assertion: For all irrational numbers a, b with $a + b \neq 0$, the number $a/(a+b)$ is irrational.

    This assertion is false. To disprove it, it is sufficient to find a counterexample. Choose $a = b = \sqrt{2}$. Then $a/(a+b) = 1/2$ by basic algebra, and this is rational. q.e.d.

11. Prove the following by induction.

    Theorem: $\sum_{n=0}^{k-1} 3n(n+1) = k^3 - k$.

    Proof: Let $P(k)$ be the assertion that the theorem holds for k. We prove this for all integers $k > 0$, by induction on k.

    - Base case: k=0. The sum is empty when $k = 0$ and by convention, the empty sum has value 0. Likewise, $0^3 - 0 = 0$. So $P(0)$ is true.
    - Induction step: Assume $k \geq 0$, $k \in \mathbb{Z}$ and that $P(k)$ is true. We must prove $P(k+1)$ holds.

      The summation is equal to

$$\begin{aligned}
\sum_{n=0}^{k} 3n(n+1) &= \sum_{n=0}^{k-1} 3n(n+1) + 3k(k+1) \\
&= k^3 - k + 3k^2 + 3k && \text{by the ind. hyp.}
\end{aligned}$$

The righthand side is equal to

$$(k+1)^3 - (k+1) \;=\; k^3 + 3k^2 + 3k + 1 - k - 1.$$

These are clearly equal, so $P(k+1)$ holds.

Q.E.D.