Math 15A - Discrete Mathematics - Spring 1999

Midterm Exam — May 6 — Answer Key

You may use a *single* sheet of notes (double-sided, $8\frac{1}{2} \times 11$ paper). You may not use the textbook or a calculator. There are 10 problems worth a total of 360 points. You have one hour and 15 minutes.

Before you begin, write your name & ID number on the cover page, and check that you have all ten problems in your exam.

Good luck!

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Student ID#:

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1. (30 points) For each formula in the left column, find a logically equivalent formula in the right column.

c	$(p \rightarrow q) \vee (q \rightarrow p)$	**a.** p
e	$(p \rightarrow q) \wedge (q \rightarrow p)$	**b.** $\sim p$
c	$p \rightarrow (q \rightarrow p)$	**c.** $p \rightarrow p$
f	$\sim(q \rightarrow \sim p)$	**d.** $p \wedge \sim p$
g	$\sim(\sim p \rightarrow q)$	**e.** $p \leftrightarrow q$
		f. $p \wedge q$
		g. None of the above.

2. (24 points) Convert the propositional formula

$$p \rightarrow (q \rightarrow (r \leftrightarrow s)).$$

into an equivalent formula which uses only the connectives $\wedge$, $\vee$ and and $\sim$.

ANSWER: $\sim p \vee \sim q \vee (r \wedge s) \vee (\sim r \wedge \sim s)$.
OTHER ANSWERS ARE POSSIBLE.

3. (36 points) Draw a Boolean circuit, using only AND, OR and NOT gates, which has three inputs x, y, and z and which outputs 1 (True) if and only if exactly one of its three inputs has value 1.

ANSWER: I'll give you the formula: from this it is easy to draw the circuit (by hand, if not by computer).

OR(AND(x,NOT(y),NOT(z)),AND(NOT(x),y,NOT(z)),AND(NOT(x),NOT(y),z))

4. (42 points) Let $T(x,y)$ be the predicate "x trusts y". Let $Y(x)$ be the predicate "x is young", and $A(x)$ be the predicate "x is aged". Let quantified variables in $(\forall x)$ and $(\exists y)$ range over all people. Express the following formally, using quantifiers, Boolean connectives and the three predicates $T(\cdot,\cdot)$, $Y(\cdot)$, $A(\cdot)$ (but do not use symbols for the set of young or aged people):

   (a) Everyone trusts someone.
   ANS: $\forall x \exists y T(x,y)$

   (b) No one trusts everyone.
   ANS: $\forall x \exists y \sim T(x,y)$

   (c) There is person who is trusted by no one.
   ANS: $\exists x \forall y \sim T(y,x)$

   (d) There is person who is trusted by no young people..
   ANS: $\exists x \forall y (Y(y) \rightarrow \sim T(y,x))$

   (e) There is a person who is trusted by everyone, but who trusts no one.
   ANS: $\exists x ((\forall y T(y,x)) \wedge (\forall z \sim T(x,z)))$

   (f) Some aged people trust no one.
   ANS: $\exists x (A(x) \wedge \forall y \sim T(x,y))$

   (g) Some young people trust all aged people.
   ANS: $\exists x (Y(x) \wedge \forall y (A(y) \rightarrow T(x,y)))$

5. (36 points) Express the following assertions (semi-)formally. You may use quantifers such as "$\forall$ rational x", "$\exists$ positive integer y", Boolean connections, and predicates such as "$x < y$", "$x \leq y$", "$x = y$" "$x \neq y$", $x \in \mathbb{Z}$, and the functions addition, subtraction, multiplication, division and square roots. hfill Hints: These assertions should be expanded out into the appropriate definitions. Do not forget to include any necessary conditions such as "x is an integer".

   (a) x divides y.
   ANS: $\exists$ integer q s.t. $(x \cdot q\ y)$ (You may add $x = 0$ if you wish.)

   (b) r is equal to $(x \bmod d)$.
   ANS: $0 \leq r < d$ and $\exists$ integer q s.t. $x = q\dot{d} + r$.

   (c) n equals $\lfloor x \rfloor$.
   ANS: $n \leq x < n + 1$.

   (d) x is a prime.
   ANS: $x > 1$ and $\forall$ positive integers s, r (if $s \cdot r = x$ then $s = 1$ or $r = 1$).

6. (36 points) (a) Let n be the integer with binary (base 2) representation 110101. Express n in base 10:

   ANSWER: 53

   (b) Let m be the negative integer which has 8-bit, two's-complement, binary representation 11110110. Express m in base 10.

   ANSWER: -10

7. (36 points) Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the values $a_0 = 1$ and $a_{n+1} = \sqrt{2} \cdot a_n$ for all $n \geq 0$.

   To answer the next questions, you should give exact expressions, including the $\sqrt{2}$ as necessary. (Do not use a calculator to approximate in decimal form.)

   (a) What is the value of a_{10}? ANSWER: $(\sqrt{2})^1 0 = 2^5 = 32$.

   (b) What is the value of $\sum_{i=0}^{11} a_i$? ANSWER: $\frac{\sqrt{2}^1 2-1}{\sqrt{2}-1} = \frac{63}{\sqrt{2}-1} = 63(\sqrt{2}+1)$.

The next three problems ask you to give proofs. Your answers will be graded on form as well as content: be sure to explain all your steps, use correct grammar, and follow the conventions for writing proofs.

8. (40 points) Either prove or disprove the following assertion:

   Assertion: For all integers n, $n^3 - n$ is divisible by 3.

   ANSWER: This true. The most straightforward proof uses a proof by cases based on n mod 3.

   Proof: Let n be an integer. By basic algebra, it will suffice to prove that n^3 mod 3 is equal to n mod 3. This is because if $n = 3k + r$ and $n^3 = 3 \cdot m + r$ for some integers k and m, then $n^3 - n = 3(m - k)$ and is thus divisible by 3.

   Case 1: $n = 3k$ for some integer k. Then $n^3 = 3(9k^3)$ is also a multiple of 3.

   Case 2: $n = 3k + 1$ for some integer k. Then $n^3 = 3(3^2k^3 + 3^2k^2 + 3k) + 1$ by basic algebra. So n^3 mod $3 = n$ mod 3.

   Case 3: $n = 3k+2$ for some integer k. Then $n^3 = 3(3^2k^3+2\cdot3^2k^2+2^2\cdot3k+2)+2$ by basic algebra. So n^3 mod $3 = n$ mod 3.

   Q.E.D.

9. (40 points) Either prove or disprove the following assertion:

   Assertion: For all rational numbers a and b, a/b is a rational number.

   ANSWER: This is false. Taking $b = 0$ and a any rational number yields a counterexample.

10. (40 points) Prove by induction on n:

    Theorem: Let $a_1 = 1$ and $a_{n+1} = \frac{n^2}{n+1} a_n$ for all integers $n \geq 1$. Then, for all positive integers n, $a_n = \frac{(n-1)!}{n}$.

    Proof: We prove the assertion by induction on $n \geq 1$.

    Base case: $n = 1$. $e_1 = 1$. And $\frac{0!}{1} = \frac{1}{1} = 1$.

    Induction step: Suppose $a_n = \frac{(n-1)!}{n}$ is true. We shall prove that $a_{n+1} = \frac{n!}{n+1}$. We argue as follows:

$$
\begin{aligned}
a_{n+1} &= \frac{n^2}{n+1} \cdot a_n \\
&= \frac{n^2}{n+1} \cdot \frac{(n-1)!}{n} \qquad \text{by the ind. hyp.} \\
&= \frac{n!}{n+1} \qquad \text{by basic algebra.}
\end{aligned}
$$

    Q.E.D.