Math 155B — Computer Graphics — Spring 2020
Homework #3 — Due Friday, April 10, 9:00pm
Hand in via Gradescope

1. A degree one Bézier curve $\mathbf{q}(u)$ has control points $\mathbf{p}_0 = \langle 0,0,0\rangle$ and $\mathbf{p}_1 = \langle 1,2,4\rangle$. Re-express this as a degree two Bézier curve. Also, reexpress it is as degree three Bézier curve.

2. Consider the curve consisting of the portion of the parabola $y = x^2$ between $\langle 0,0\rangle$ and $\langle 2,4\rangle$. First express this portion as a degree 2 Beźier curve by giving its three control points. Second, express this as a degree 3 Bézier curve by giving its four control points.

3. The GLSL function `smoothstep` computes a generalized version of the degree 3 Hermite function $H_3(u)$. The function H_3 is a real-valued, cubic curve with $H_3(0) = H_3'(0) = H_3'(1) = 0$ and $H_3(1) = 1$. Express $H_3(u)$, for $u \in [0,1]$, as a degree 3 Bézier curve in $\mathbb{R}$ by giving its four control points. (Note these control points lie in $\mathbb{R}$.)

4. Express the derivative of $H_3(u)$ (problem 3) as a degree 2 Bézier curve.

5. For the curve $H_3(u)$ of problem 4, describe the convex hull of the control points of the Bézier curve. What does this tell you about the values of $H_3(u)$? What does the variation diminishing property, applied to the lines $y = y_0$ (for y_0 constant), tell you about the graph of the function $H_3(u)$?

6. (Reversing degree elevation.) Suppose you are given the control points $\mathbf{p}_0, \ldots, \mathbf{p}_3$ for a degree Bézier curve $\mathbf{q}(u)$ (working in ordinary coordinates, not homogeneous coordinates). Also suppose that it is promised that $\mathbf{q}(u)$ is actually a *degree 2* polynomial curve. How would you compute the control points representing $\mathbf{q}(u)$ as a degree 2 Bézier curve?

Suppose now that $\mathbf{q}(u)$ is only *close to* a degree 2 curve. How would you check that this holds? Suggest a robust method for converting $\mathbf{q}(u)$ into a degree 2 Bézier curve $\mathbf{q}^*(u)$ which is "close to" $\mathbf{q}(u)$. How would your method compute the control points for $\mathbf{q}^*(u)$? Can you give a reasonable and easy-to-compute upper bound on the maximum difference $||\mathbf{q}^*(u) - \mathbf{q}(u)||$ between the two curves for $u \in [0,1]$?

7. Fill in the details of the following sketch of a proof of the variation diminishing property of Bézier curves. First, fix a line (or, in $\mathbb{R}^3$, a plane) and a continuous curve (the curve may consist of straight line segments). Consider the following operation on the curve: choose two points on the curve and replace the part of the curve between the two points by the straight line segment joining the two points. Prove that this does not increase the number of times the curve crosses the line. Second, show that the process of going from the control polygon of a Bézier curve to the two control polygons of the two subcurves obtained by using recursive subdivision to split the curve at $u = 1/2$ involves only a finite number of uses of the operation from the first step. Therefore, the total number of times the two new control polygons cross the line is less than or equal to the number of times the original control polygon crossed the curve. Third, prove that as the curve is repeatedly recursively subdivided, the control polygon approximates the curve. Fourth, argue that this suffices to prove the variation diminishing property (this last point is not entirely trivial).

8. Give a full acknowledgement of assistance. This includes anyone or any web site, etc., that helped you; and anyone you helped.