4. [10 points] Consider the yaw-pitch-roll values $y = 0$, $p = 90°$, and $r = -90°$. Express this orientation as a 3×3 rotation matrix.

1. [15 points] For each of the following, state whether it is true or false by writing "T" or "F" on the line. Then, if it is false, give a counterexample. (In order to be true, it must be true for *all* quaternions q_1, q_2.)

______**a.** $q_1 q_2 = q_2 q_1$.

______**b.** $q_1 q_2 = \pm q_2 q_1$. (That is, either they are equal, or they are opposite in sign.)

______**c.** $q_1^* q_1 = q_1 q_1^*$.

______**d.** $q_1^* q_2 = q_2^* q_1$.

______**e.** $q_1 + q_1^*$ is a scalar. (That is, with no i, j, k components)

______**f.** $q_1^* q_1$ is a scalar.

2. [10 points] Compute the following:

a. $(1 + j + k)^{-1}$.

b. $(1 + j + k)^*$.

c. $||1 + j + k||$.

3. [25 points] Let $q_1 = i + k$.

a. What rotation $R_{\theta,\mathbf{u}}$ does q_1 represent? Express your answer with $\mathbf{u}$ a unit vector.

b. Find a quaternion q_2 such that $q_1 = (q_2)^2$. (So q_2 is a square root of q_1.)

c. Find a quaternion q_3 such that $q_1 = (q_3)^3$. (So q_3 is a cube root of q_1.)

d. In part (b), you found a square root q_2 of q_1. Of course, $-q_2$ is also a square root. Give two more square roots of q_1.

5. [20 points] Conversion of yaw-pitch-roll to rotation matrices and to quaternions. Convention: x-axis is leftward, y-axis is upward, and z-axis is forward.

a. Let Yaw be $-90°$, Pitch be $90°$ and Roll be $-90°$. Express this orientation as (i) a rotation matrix, and (ii) a quaternion.

b. Let Yaw be $-90°$, Pitch be $90°$ and Roll be $90°$. (Change in sign for Roll.) Express this orientation as (i) a rotation matrix, and (ii) a quaternion.

5. [20 points] Consider the yaw-pitch-roll values $y_1 = 90°$, $p_1 = -90°$, and $r_1 = 0$.

a. Find **two** other yaw-pitch-roll values which represent the same orientation.

b. Are there yaw-pitch-roll values q_2, p_2, r_2 with $p_2 = 90°$ which represent the same orientation as y_1, p_1, r_2? If so, give one example.

6. [15 points] Let $\mathbf{x} = \langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ and $\mathbf{y} = \langle 1, 0, 0 \rangle$.

a. What is $\textsc{Lerp}(\mathbf{x}, \mathbf{y}, \frac{1}{3})$ equal to?

b. What is $\textsc{Slerp}(\mathbf{x}, \mathbf{y}, \frac{1}{3})$ equal to?

6. [28 points] Answer the following questions about quaternions.

a. Compute $(1 + j)^2$.

b. Compute $(1 + j)(i + k)$.

c. Compute $(i + k)^{-1}$.

d. Give an example of two unit quaternions such that $q_1 q_2 \neq q_2 q_1$.

e. Let q be the unit quaternion $q = \frac{\sqrt{3}}{2} i + \frac{1}{2} k$. This defines a rotation $R_{\theta,\mathbf{u}}$. Give the values of θ and $\mathbf{u}$.

f. Now let q be the unit quaternion $q = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2\sqrt{2}} i + \frac{1}{2\sqrt{2}} k$. This also defines a rotation $R_{\theta,\mathbf{u}}$. Give its values of θ and $\mathbf{u}$.

7. [10 points] A quaternion $q = d + ai + bj + ck$ defines a rotation R in $\mathbb{R}^3$. Give the formula for $R(\mathbf{k})$, that is, the result of applying the rotation to $\mathbf{k} = \langle 0, 0, 1 \rangle$.

9. [10 points] Let $\vec{u} = \langle 1, 0, 0 \rangle$ and $\vec{v} = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}} \rangle$ be unit vectors in $\mathbb{R}^3$.

Compute $\textsc{Slerp}(\frac{1}{3}, \mathbf{u}, \mathbf{v})$.

1. A airplane is flying northward, oriented horizontally as usual. This is the orientation for yaw, pitch and roll all equal to zero.

a. The airplane changes to flying south and flipped upside down. Characterize this new orientation in terms of yaw, pitch and roll.

b. Give another, essentially different, yaw-pitch-roll characterization of the orientation of part a. "Essentially different" means not just adjusting angles by a multiple of 360 degrees.

Quaternions may be written as either $\langle d, a, b, c\rangle$ or $d + ai + bj + ck$. You may use arctan, arcsin or arccos in your answers **if** necessary.

For these problems, let q_1, q_2 and q_3 be the quaternions $q_1 = \langle 1, 0, 0, 0\rangle$, $q_2 = \langle 0, 0, 1, 0\rangle$ and $q_3 = \langle 2, 0, 0, 1\rangle$.

1. What are q_1^*, $||q_1||$ and q_1^{-1}?

2. What are q_2^*, $||q_2||$ and q_2^{-1}?

3. What are q_3^*, $||q_3||$ and q_3^{-1}?

4. What rotation $R_{\theta,\mathbf{v}}$ is represented by q_1? (Give θ and $\mathbf{v}$. For these three questions, multiple answers are possible.)

5. What rotation $R_{\theta,\mathbf{v}}$ is represented by q_2?

6. What rotation $R_{\theta,\mathbf{v}}$ is represented by q_3?

1. For each of the following, answer "T" (True) if the equation holds for all quaternions; and answer "F" if the equation is (sometimes) false.

____ a. $q_1 q_2 = q_2 q_1$

____ b. $q_1 + q_2 = q_2 + q_1$

____ c. $q_1^* q_1 = q_1 q_1^*$

____ d. $q_1^{-1} q_1 = 1.$

____ e. $(q^*)^* = q$

____ f. $q_1^* q_2 = (q_2^* q_1)^*$

____ g. $(q_2^{-1} q_1)^{-1} = (q_1^{-1} q_2)$

____ h. $(q_2^{-1} q_1)^{-1} = (q_2^{-1} q_1)$

____ i. $j^{-1} = -j.$

____ j. $ik = ki.$

2. Suppose q_1 and q_2 are quaternions that represent rotations Q and R. Let S be the rotation which is the composition of Q and R so $S(\mathbf{x}) = Q(R(\mathbf{x}))$. What quaternion represents the rotation S?

3. What quaternion represents the orientation corresponding to Yaw 90°, and Pitch −90°, and Roll −90°? (Recall that yaw-pitch-roll has the y axis pointing forwards, the x-axis pointing leftwards, and the z axis pointing upwards.)

1. Give the rotation $R_{\theta,\mathbf{u}}$ which corresponds to yaw = 90°, pitch = 0°, and roll = 0°. (For your answers to questions 1 and 2, give the values of θ and $\mathbf{u}$.)

2. Give the rotation $R_{\theta,\mathbf{u}}$ which corresponds to yaw = 0°, pitch = 0°, and roll = 45°.

3. Let Ω denote the orientation yaw = 180°, pitch = 0°, and roll = 180°. Give a second representation of Ω in terms of different yaw, pitch, roll values, with yaw equal to zero.

1. Evaluate the following quaternion products:

a. $jk =$

b. $kj =$

c. $(1 + 2j)(3i + 4k) =$

d. $kji =$

2. Let $q = 3i + 4k$.

a. What is its norm, $||q||$?

b. What is its conjugate, q^*?

c. What is its inverse, q^{-1}?

3. Give two unit quaternions which represent the rotation $R_{\pi,\mathbf{k}}$.

4. What rotation $R_{\theta,\mathbf{u}}$ is represented by the quaternion $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$? (Give θ and $\mathbf{u}$.)