Math 155A - Computer Graphics - Spring 2022
Midterm #1 — April 28

Name: ..........................................................

PID: ..........................................................

You must write your initials ON EVERY PAGE. — DO THIS FIRST! This is in case, exams pages get separated during the scanning process. At the same time, please check your copy of the midterm has all eight problems.

You have 80 minutes. There are 8 problems. You may not use calculators, notes, textbooks, computers, phones, or other resources during this exam.

All answers should be written on the same page as the problem that it answers. Please do not hand in extra pages. Please show enough of your work so that we can see how you obtained your answers.

Use a dark pencil, or a dark blue or black pen so that the test will scan legibly.

Good luck!
1. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be the affine transformation that transforms the standard “F” as shown:

\[
\begin{align*}
\langle 0, 1 \rangle & \mapsto \langle 1, 1 \rangle \\
\langle 1, 0 \rangle & \mapsto \langle 0, -1 \rangle \\
\langle 0, -1 \rangle & \mapsto \langle 1, 0 \rangle \\
\end{align*}
\]

\( f(x, y) \) is:

\[
\begin{align*}
\langle 0, 1 \rangle & \mapsto \langle 2, -1 \rangle \\
\langle 1, 0 \rangle & \mapsto \langle 1, -1 \rangle \\
\langle 0, -1 \rangle & \mapsto \langle 1, 0 \rangle \\
\end{align*}
\]

(a) Is \( f \) linear? If not, give an example of how it fails to be linear.

(b) Give a 3 \times 3 matrix \( M \) that represents \( f \) over homogeneous coordinates.

(c) Give a 3 \times 3 matrix \( M \) that represents \( f^{-1} \) over homogeneous coordinates.

(d) Is \( f \) a rigid transformation?

(e) Is \( f \) an orientation preserving transformation?
2. Recall that, working in $\mathbb{R}^2$, the generalized rotation transformation $R_u^\theta$ performs a rotation of angle $\theta$ around the point $u \in \mathbb{R}^2$. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps the “F” shape as shown:

Express $g$ in the form $R_u^\theta$, giving $\theta$ and $u$ explicitly.
3. Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that performs a reflection across the plane $P = \{ (x, y, z) : x = -2 \}$.

(a) Give a $4 \times 4$ matrix that acts on homogeneous coordinates and represents $h$.

(b) Express $h$ as a composition of scalings $S_{(a,b,c)}$, translations $T_u$, and rotations $R_{\theta,u}$. Your answer may not need to use all three kinds of transformations. For scalings $S_{(a,b,c)}$, it is permitted to have the values $a, b, c$ be negative, positive or zero.
4. Let \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by
\[
h((x, y, z)) = \left( \frac{x+1}{z+1}, 7, \frac{x-y}{z+1} \right).
\]
(a) Is \( h \) a linear transformation?
(b) Is \( h \) an affine transformation?
(c) Give a \( 4 \times 4 \) matrix that acts on homogeneous coordinates and represents the transformation \( h \).

5. Consider the region \( \mathcal{R} \) shown in gray in the figure.
(a) Express \( \mathcal{R} \) as a triangle strip by listing the vertices used in the triangle strip. The triangle strip should have front faces pointing towards the viewer.
(b) Express \( \mathcal{R} \) as a pair of triangle fans by listing the vertices used in the two triangle fans. The triangle fans should have front faces pointing towards the viewer.
6. Fix $u \in \mathbb{R}^3$ to be the vector $u = (2, 1, 0)$.

Give the $3 \times 3$ matrix that represents the linear map in $\mathbb{R}^3$ defined by

$$x \mapsto \frac{(u \cdot x)u}{||u||^2}$$
7. A light source is placed at \((0, 10, 1)\) and it casts shadows onto the plane \(P\) defined by \(y = 1\). This plane is horizontal, parallel to the \(xz\)-plane.

When \((x, y, z)\) is a point in \(\mathbb{R}^3\) with \(1 \leq y < 10\), define \(A((x, y, z))\) to be the position of the shadow of the point on the plane \(P\). For example, \(A((0, 5, 1)) = (0, 1, 1)\), and \(A((2, 4, 3)) = (1, 1, 4)\).

a. Working in ordinary coordinates (not homogeneous coordinates) give the formula expressing the mapping \(A((x, y, z)) = (x', y', z')\). That is, give formulas for \(x', y', z'\) in terms of \(x, y, z\).

b. Give a \(4 \times 4\) matrix that represents the transformation \(A\) over homogeneous coordinates.
8. (The Painter’s Algorithm.)
(a) Briefly explain the Painter’s algorithm. What is its purpose? Describe the algorithm. What are its advantages and disadvantages compared with other algorithms for the same purpose?

(b) Describe situations where it might be helpful to use the Painter’s algorithm, or methods similar to the Painter’s algorithm, in conjunction with the Depth Buffer method.