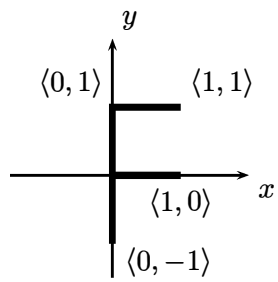
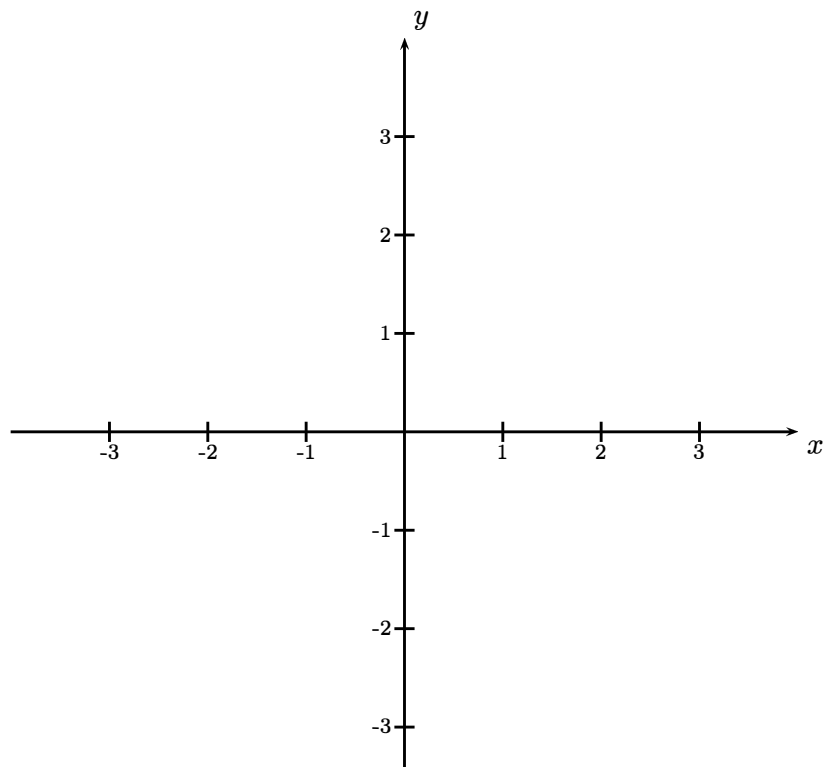
**1.** [20 points] Consider the following $3 \times 3$ matrix $M$ representing a transformation in $\mathbb{R}^2$ over homogeneous coordinates.

**a.** Draw the image of the "F" under this transformation on the large axes to the right. Be sure to label enough points to make your answer clear.

$$M = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



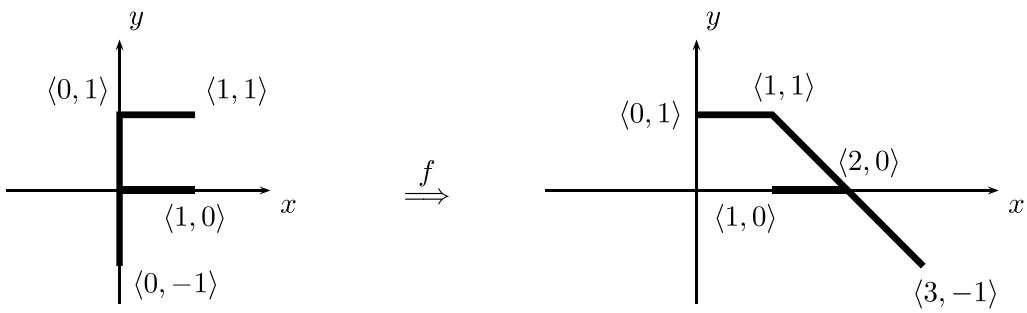




**b.** Is this transformation rigid? Is it orientation preserving?

**c.** Express the transformation as a generalized rotation $R_\varphi^{\mathbf{v}}$ by giving $\mathbf{v}$ and $\varphi$ — or explain why this is not possible. (Recall that a generalized rotation $R_\varphi^{\mathbf{v}}$ is a rotation around $\mathbf{v}$.)

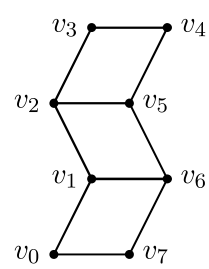
**d.** Express the transformation as a composition of zero or more rotations $R_\theta$, scalings $S_{\mathbf{u}}$, and translations $T_{\mathbf{u}}$. (Do not give matrices.)

**2.** [10 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.




Give a $3 \times 3$ matrix which represents $f$ over homogeneous coordinates.

**3.** [10 points] Eight vertices are used to specify the three quadrilaterals as shown. List the vertices in the correct order to render the quadrilaterals as a single triangle strip. List the vertices so that the faces are facing towards the viewer as shown (when using the default conventions for front faces).



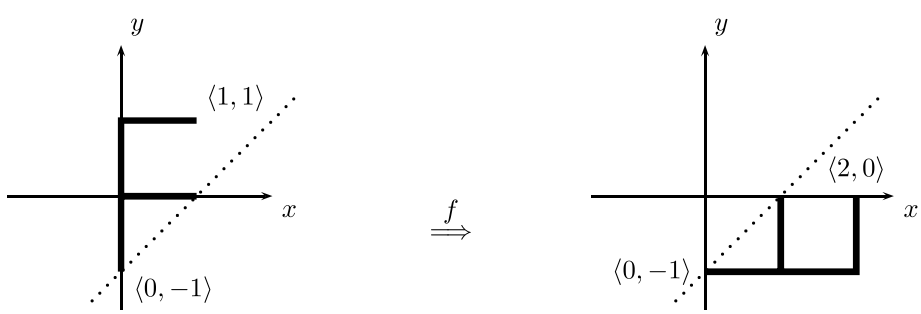
**6.** [10 points] State the definition of "affine transformation".

**7.** [10 points] Describe the depth buffer method for hidden surfaces. What are its advantages? What are its disadvantages?

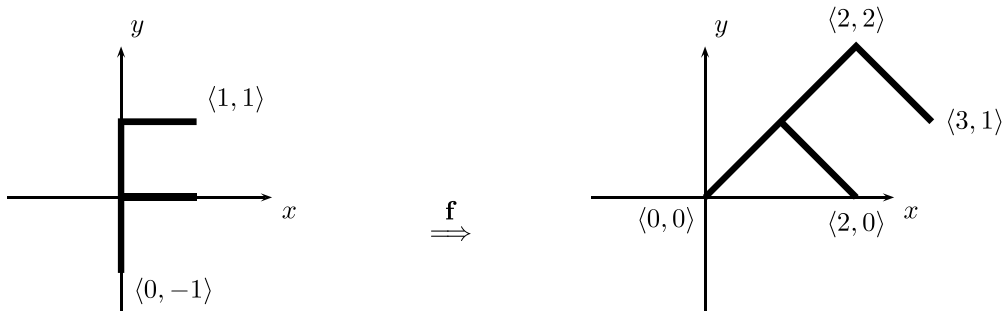
**3.** [10 points] Give the definition of **linear transformation**.

**4.** [20 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined to be the transformation that reflects points across the line $y = x - 1$. In particular, it maps a "F" as shown in the picture below.

Give the matrix that represents $f$ over homogeneous coordinates.



**1.** [20 points] Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined to be the affine transformation that maps an "F" as shown in the picture below.




**a.** Is $f$ a rigid transformation? Explain why or why not.

**b.** Express $f$ in the form $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$ with $M$ a $2 \times 2$ matrix.

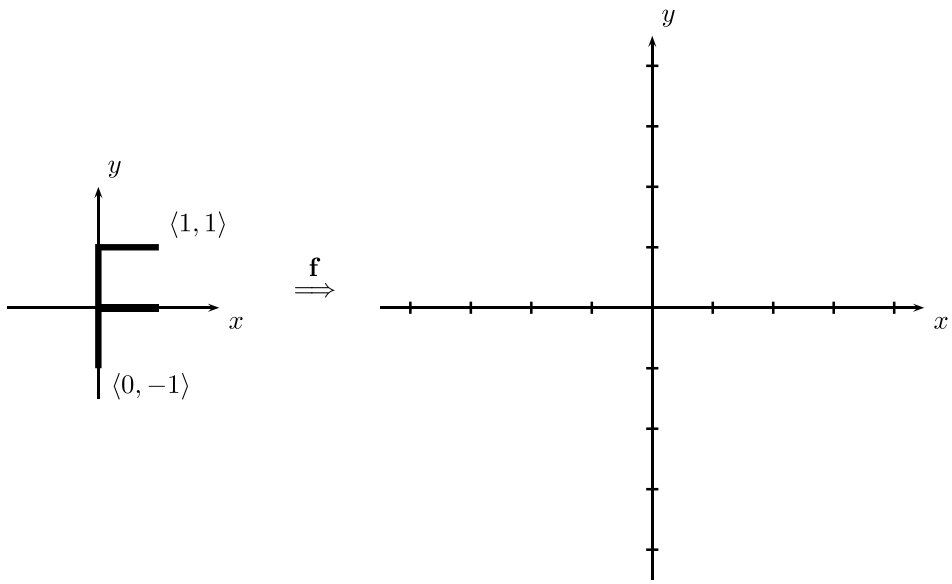
**c.** ~~Give a sequence of "pseudo" OpenGL commands that will draw the "F" in the position shown on the right. Use commands such as `drawF()` (draws "F" in the position shown on the left), `pglRotatef(···)`, `glTranslate2f(···)`, `glLoadIdentity()`, and `pglScale2f(···)`.~~

Express the transformation $f$ as a composition of transformations of the forms $R_\theta$, $T_{\vec{u}}$, and $S_{\langle a,b\rangle}$.

**2.** [15 points] Consider the following $3 \times 3$ matrix $M$ that operates on the homogeneous coordinates of points in $\mathbb{R}^2$.

$$\begin{pmatrix} -2 & -2 & 2 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

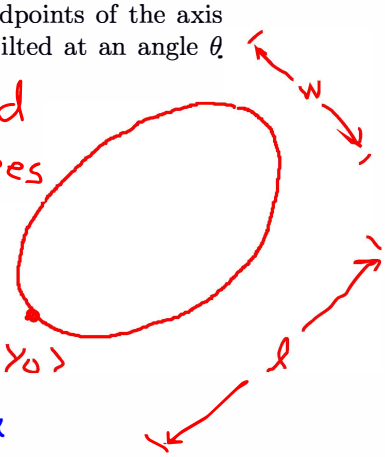
In the empty graph on the right, draw the image of the "F" under the affine map on $\mathbb{R}^2$ that is defined by the matrix $M$. Draw to scale, and label points as needed.



**1.** [20 points] This problem concerns transformations in $\mathbb{R}^2$. Suppose you are given a function `DrawCircle()` that draws a unit circle centered at the origin (radius equals one). Give a code fragment that will draw an ellipse as shown in the figure. The length of the ellipse is $\ell$ and the width is $w$. One endpoint of the ellipse is at $\langle x_0, y_0 \rangle$ in $\mathbb{R}^2$, namely, one of the endpoints of the axis along which the length $\ell$ is measured. The ellipsoid is tilted at an angle $\theta$ ~~(measured in degrees)~~.

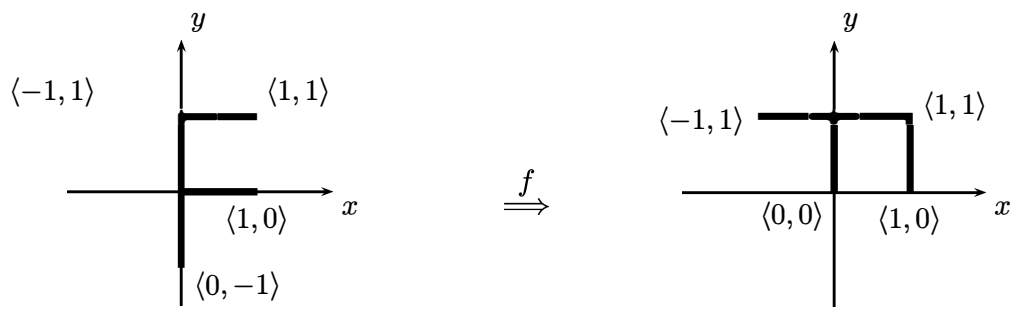
~~Your code fragment that draws the ellipse may use any of the following pseudo OpenGL commands. glMatrixMode();~~ ~~glLoadIdentity()~~, ~~pglRotatef~~(), ~~pglTranslatef()~~, pglLoadMatrix~~()~~, pgl~~MultMatrix()~~, ~~pglScaTef()~~, and ~~DrawCircle()~~.

Tilted $\theta$ degrees ccw




Describe the 3x3 matrix which will transform the unit circle centered at the origin to be ellipse as pictured. Describe the matrix as a composition of rotations $R_\theta$, translations $T_{\vec{u}}$ and scalings $S_{\langle a,b \rangle}$.

**1.** [36 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.




**a.** Express $f$ in the form $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$ with $M$ a $2 \times 2$ matrix.

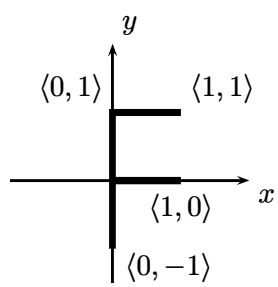
**e.** Now consider the inverse $f^{-1}$ of the transformation $f$. Give a $3 \times 3$ matrix $N$ that represents $f^{-1}$ in homogeneous coordinates.

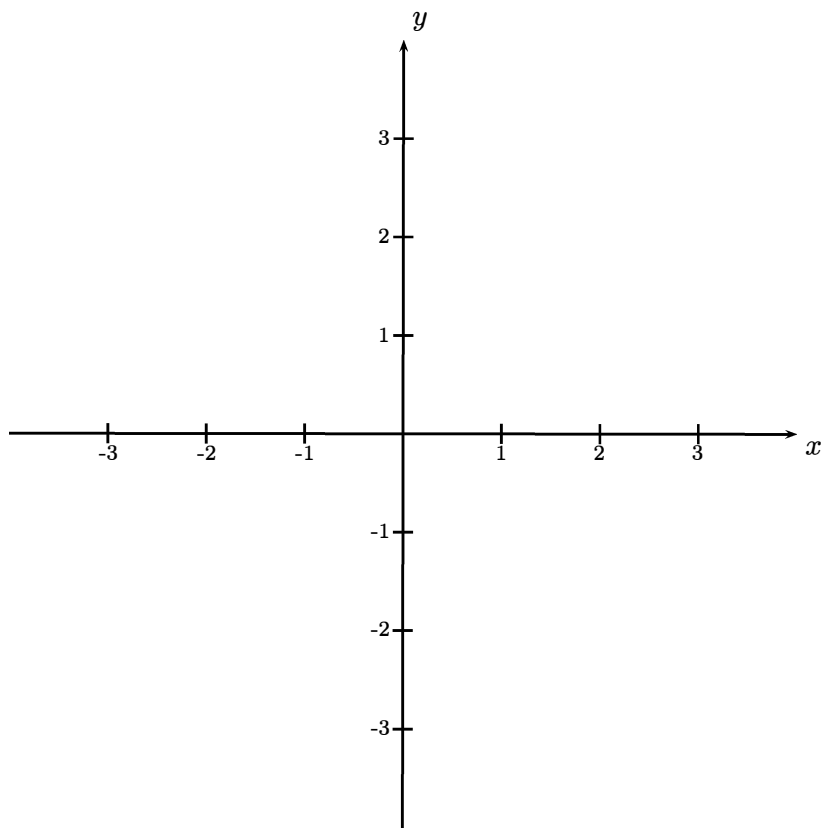
**f.** Express $f$ as a generalized rotation $f = R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ by giving the rotation angle $\theta$ and the center point $\mathbf{u}$ of the generalized rotation, or explain why this is not possible

**1.** [10 points] Recall that a generalized rotation $R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ is the rigid orientation-preserving transformation which rotates counterclockwise around the point $\mathbf{u}$ (holding the point $\mathbf{u}$ fixed. Express $R_\theta^{\mathbf{u}}$ as a composition of rotations $R_\varphi$ and translations $T_{\mathbf{v}}$.

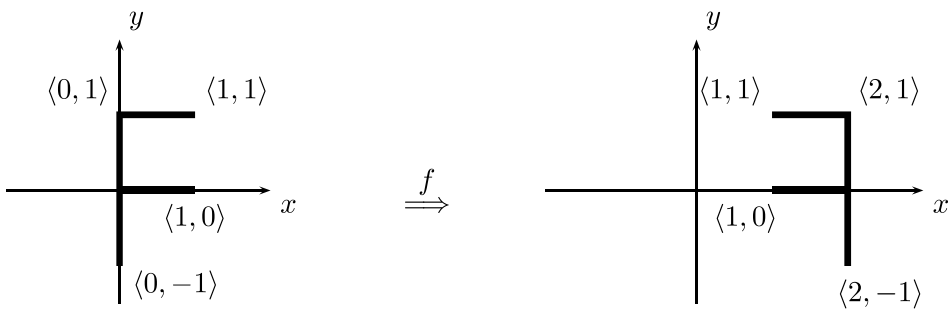
**2.** [10 points] Consider the following $3 \times 3$ matrix $M$ representing a transformation in $\mathbb{R}^2$ over homogeneous coordinates. (Watch out for the lower right entry!) Draw the image of the "F" under this transformation on the large axes to the right. Be sure to label enough points to make your answer clear.

$$M = \begin{pmatrix} 0 & 2 & -2 \\ -4 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$



**3.** [40 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.




**a.** Give a $3 \times 3$ matrix which represents $f$ over homogeneous coordinates.

**b.** Now consider the inverse $f^{-1}$ of the transformation $f$. Give a $3 \times 3$ matrix $N$ that represents $f^{-1}$ in homogeneous coordinates.

**c.** Express $f$ as a composition of rotations $R_\theta$, translations $T_{\mathbf{u}}$, and/or scalings $S_{\langle a,b\rangle}$, or explain why this is not possible.

**d.** Express $f$ as a generalized rotation $f = R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ by giving the rotation angle $\theta$ and the center point $\mathbf{u}$ of the generalized rotation, or explain why this is not possible

**1.** [20 points] Consider the following OpenGL commands:

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
glRotatef( 90.0, 0.0, 1.0, 0.0 );
glTranslatef( 2.0, 0.0, 0.0 );
glScalef( 2.0, 1.0, 1.0 );
```

What will the $4 \times 4$ modelview matrix be equal to after these commands have executed?

Consider the transformation

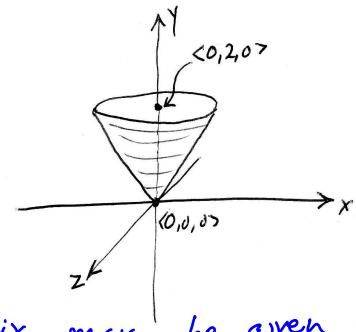
$$R_{\frac{\pi}{2}, \langle 0,1,0 \rangle} \circ T_{\langle 2,0,0 \rangle} \circ S_{\langle 2,1,1 \rangle}$$

Give the 4x4 matrix which represents this transformation over homogeneous coordinates.

**1.** [20 points] This problem concerns transformations in $\mathbb{R}^3$. Suppose you are given a function `DrawCone()` that draws a cone of height 1, and base radius 1. This cone drawn by `DrawCone()` is situated centered around the $y$-axis with its base on the $xz$ plane and the tip of the cone at $\langle 0, 1, 0\rangle$.

**a.** Give a ~~code fragment~~ description of a transformation that will draw the cone as shown in the figure: the cone is to be drawn upside down, and with height 2 and base radius 2. Its tip is now at the origin; it is still centered around the $y$-axis.

~~Your code fragment that draws the cone may use any of the following pseudo OpenGL commands: `glMatrixMode()`, `glLoadIdentity()`, `glRotatef()`, `glTranslatef()`, `glLoadMatrix()`, `glMultMatrix()`, `glScalef()`, and `DrawCone()`.~~




Your description of the matrix may be given as a composition of rotations ($R_{\theta,\vec{u}}$), translations ($T_{\vec{u}}$) and scalings ($S_{\langle a,b,c\rangle}$).

**b.** Give a $4 \times 4$ homogeneous matrix that gives the same transformation as is used in your answer for part a.

**3.** [12 points] Suppose the function `drawTwoPoints()` draws a point at $\langle 0, 0, 0\rangle$ and another point at $\langle 1, 1, 0\rangle$.

**a.** Consider the ~~sequence of OpenGL commands:~~ transformation A defin

```
glLoadIdentity();
glTranslatef(-1, 0, 0);
glScalef(2, 1, 2);
glRotatef(90, 0, 1, 0);
drawTwoPoints();
```

$T_{\langle -1,0,0\rangle} \circ S_{\langle 2,1,2\rangle} \circ R_{\pi/2,\langle 0,1,0\rangle}$

When the drawTwoPoints() is called, where does the point it draws at $\langle 0, 0, 0\rangle$ actually get placed (as transformed by the ModelView matrix)? And, where does the point it draws at $\langle 1, 1, 0\rangle$ get placed?

When transformed by A, where do these two points lie?

**b.** Now consider the slightly different ~~sequence of OpenGL commands.~~

transformation B:

```
glLoadIdentity();
glRotatef(90, 0, 1, 0);
glScalef(2, 1, 2);
glTranslatef(-1, 0, 0);
drawTwoPoints();
```

$R_{\frac{\pi}{2},\langle 0,1,0\rangle} \circ S_{\langle 2,1,2\rangle} \circ T_{\langle -1,0,0\rangle}$

When the drawTwoPoints() is called, where does the point it draws at $\langle 0, 0, 0\rangle$ actually now get placed? (You only need to answer about this one point.)

Where are the two points placed by B?

**4.** [10 points] Suppose we are modelling a Solar System and need to define a transformation $A$ which will place the Earth in the right position. The sun is at the origin. The Earth is distance $d$ from the Sun, lying in the $xz$ plane. We wish to revolve the Earth by angle $\theta$ around the Sun (for the time of year). We wish to rotate the Earth on its axis by angle $\varphi$ (for time of day). We wish to draw the Earth with radius $r$. (There is no tilt!)

Suppose we have a routine that draws the Earth as a radius 1 sphere centered at the origin. What transformation $A$ needs to be used to place the Earth as desired? Express your answer $A$ as a composition of rotations $R_{\psi,\vec{u}}$. translations $T_{\mathbf{u}}$, and scalings $S_{a,b,c}$.

**3.** [20 points] A light source is placed at the origin in $\mathbb{R}^3$, and it casts shadows onto the plane defined by $z = -10$. Thus, the plane is like an infinite wall parallel to the $xy$-plane, placed at $z = -10$.

For $\mathbf{x} = \langle x_1, y_1, z_1 \rangle$ a point in $\mathbb{R}^3$ where $z_1 < 0$, let $A(\mathbf{x}) = \langle x_2, y_2, z_2 \rangle$ be the point on the wall where the shadow of $\mathbf{x}$ is. This means that $z_2 = -10$. Give a $4 \times 4$ matrix that represents the transformation $A$ over homogenous coordinates, **or**, prove that there is no such matrix.

**5.** [20 points] A light source is placed at $\langle -10, 0, 0 \rangle$ and it casts shadows onto the $yz$-plane $P$ defined by $x = 0$. The $yz$-plane is like an infinite wall.

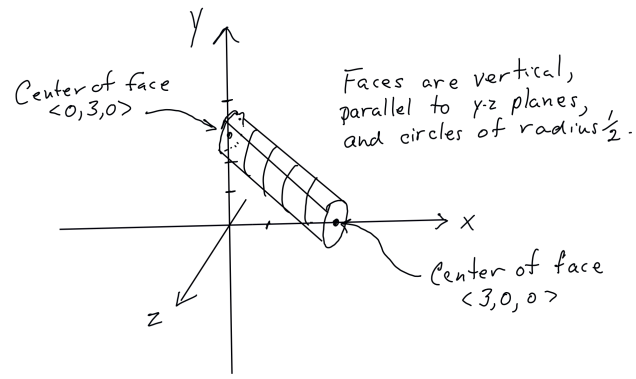
When $\langle x, y, z \rangle$ is a point in $\mathbb{R}^3$ with $-10 < x \leqslant 0$, define $A(\langle x, y, z \rangle)$ to be the position of the shadow of the point on the $yz$-plane. For example, $A(\langle -5, 1, 2 \rangle) = \langle 0, 2, 4 \rangle$, and $A(\langle -8, 1, 2 \rangle) = \langle 0, 5, 10 \rangle$

**a.** Working in ordinary coordinates (not homogeneous) give the formula expressing the mapping $A(\langle x, y, z \rangle) = \langle x', y', z' \rangle$. That is, give formulas for $x', y', z'$ in terms of $x, y, z$.

**b.** Give a $4 \times 4$-matrix that represents the transformation $A$ over homogeneous coordinates.

**4.** [20 points] Suppose $\mathcal{C}$ is a radius 1, height 2 cylinder centered at the origin, with central axis the $y$-axis. The top face of $\mathcal{C}$ is the horizontal disk of radius one centered $\langle 0, 1, 0\rangle$. The bottom face of $\mathcal{C}$ is the horizontal disk of radius one centered $\langle 0, -1, 0\rangle$. ("Horizontal" means parallel to the $xz$-plane.)

Let $\mathcal{D}$ be the skewed cylinder shown in the figure which has central axis the line containing $\langle 3, 0, 0\rangle$ and $\langle 0, 3, 0\rangle$. The right face of $\mathcal{D}$ is the vertical radius $\frac{1}{2}$ disk, parallel to the $yz$ plane centered at $\langle 3, 0, 0\rangle$ and the left face of $\mathcal{D}$ the radius 1 disk lying in the $yz$-plane centered at $\langle 0, 3, 0\rangle$.




Give a $4 \times 4$ matrix $M$ which transforms $\mathcal{C}$ to $\mathcal{D}$. For full credit, keep the outward faces still outward facing after the transformation. (In other words, $M$ is orientation preserving and does not turn the cylinder inside out.)

**5.** [20 points] A light source is placed at origin and it casts shadows onto the plane $P$ defined by $x = 20$. This plane is like an infinite wall, parallel to the $yz$-plane.

When $\langle x, y, z\rangle$ is a point in $\mathbb{R}^3$ with $0<x\leqslant 20$, define $A(\langle x, y, z\rangle)$ to be the position of the shadow of the point on the plane $P$. For example, $A(\langle 5, 1, 2\rangle) = \langle 20, 4, 8\rangle$, and $A(\langle 2, 1, 2\rangle) = \langle 20, 10, 20\rangle$

**a.** Working in ordinary coordinates (not homogeneous) give the formula expressing the mapping $A(\langle x, y, z\rangle) = \langle x', y', z'\rangle$. That is, give formulas for $x', y', z'$ in terms of $x, y, z$.

**b.** Give a $4\times 4$ matrix that represents the transformation $A$ over homogeneous coordinates.

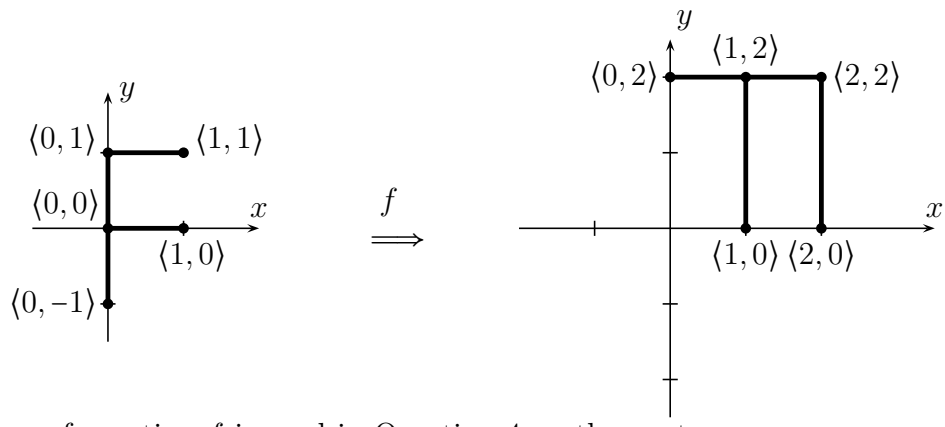
**Question 1:** Briefly describe the Painter's algorithm. Be sure to include comments on

**a.** The purpose of the Painter's algorithm.

**b.** How the Painter's algorithm works.

**c.** Disadvantages of the Painter's algorithm.

**Question 3:** An affine transformation $f$ of $\mathbb{R}^2$ maps the standard "F" shape as shown:




The same transformation $f$ is used in Question 4 on the next page.

Express $f$ as a composition of translations $T_{\mathbf{u}}$, scalings $S_{\alpha,\beta}$, and rotations $R_\theta$.

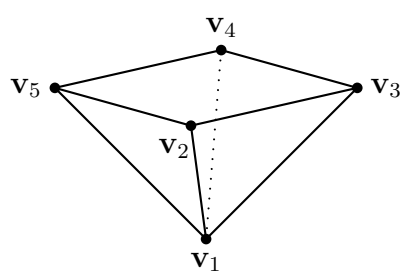
**Question 4:** Continue to work with the same transformation $f$ as in Question 3. Answer (a) and (b) below.

**(a)** Give a $3 \times 3$ matrix that represents $f$ over homogeneous coordinates.

**(b)** Give a $3 \times 3$ matrix that represents $f^{-1}$ over homogeneous coordinates.

**Question 2:** A "square cone" (or, an upside-down four-sided pyramid) has a flat, square top, and four triangular sides. Its bottom vertex is $\mathbf{v}_1 = \langle 0, 0, 0 \rangle$. Its four top vertices are

$\mathbf{v}_2 = \langle 0, 2, 2 \rangle$ (front);
$\mathbf{v}_3 = \langle 2, 2, 0 \rangle$ (right);
$\mathbf{v}_4 = \langle 0, 2, -2 \rangle$ (back); and
$\mathbf{v}_5 = \langle -2, 2, 0 \rangle$ (left).




A triangle fan is used to render the four bottom triangles of the square cone. Give the vertices — in a correct order — used for the triangle fan. How many vertices should be used in the list of vertices for the triangle fan? List the vertices in an order that makes the front faces facing outward (downward) from the cone.

**Question 1:** (Homogeneous coordinates in $\mathbb{R}^3$.)

a. What point in $\mathbb{R}^3$ is represented by the homogeneous coordinates $\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \rangle$?

b. Give three different homogenous representations for the the point $\langle 1, -2, 3 \rangle$.

**Question 2:** Let $\mathbf{u} = \langle 1, 0, 2 \rangle$. Define $f(\mathbf{x}) = \mathbf{x} \times \mathbf{u}$ ("$\times$" is vector cross product.)

a. Is $f$ a linear map?

b. Is $f$ an affine map?

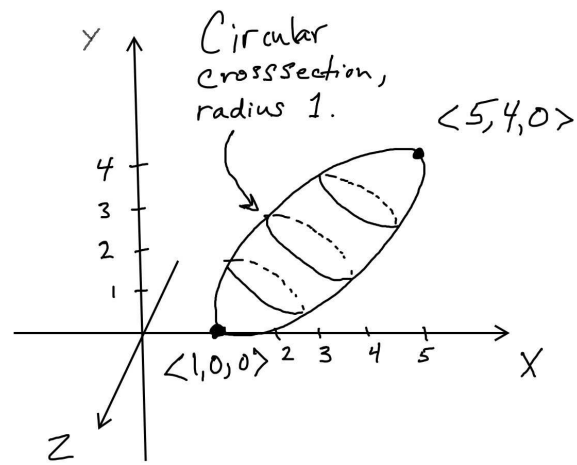
c. Give a $4 \times 4$ matrix that represents $f$ over homogeneous coordinates.

**Question 5:** A light source is placed at $\langle 10, 0, 0\rangle$ and it casts shadows onto the plane $P$ defined by $x = 2$. Note that $P$ is parallel to the $yz$ plane, and acts like an infinite wall.

When $\langle x, y, z\rangle$ is a point in $\mathbb{R}^3$ with $2 \le x < 10$, define $h(\langle x, y, z\rangle)$ to be the position of the shadow of the point on the $yz$-plane. For example, $h(\langle 4, 3, -6\rangle) = \langle 2, 4, -8\rangle$.

**a.** Working in ordinary coordinates (not homogeneous) give the formula expressing the mapping $h(\langle x, y, z\rangle) = \langle x', y', z'\rangle$. That is, give formulas for $x', y', z'$ in terms of $x, y, z$.

**For question 3 and 4:** Let $\mathcal{S}$ be the unit sphere centered at the origin. Consider an ellipsoid $\mathcal{E}$ which has circular crosssection of radius 1, and its major axis has one end at $\langle 1, 0, 0\rangle$ and the other end at the point $\langle 5, 4, 0\rangle$. We wish to define an affine transformation $g$ that maps the unit sphere $\mathcal{S}$ to the ellipsoid $\mathcal{E}$. For full credit, please give answers that make $g$ be orientation preserving.




**Question 3:** Express $g$ as a composition of rotations $R_{\theta,\mathbf{u}}$, scalings $S_{\langle a,b,c\rangle}$ and translations $T_{\mathbf{u}}$. (There are many possible answers: you only need to give one!)

**Question 4:** Give a $4 \times 4$ matrix that represents $g$ over homogeneous coordinates. (There are again multiple possible answers; your answer for b. does not need to correspond to your answer for a.