Surfaces and Normals

Last time - Two methods for computing normals.

Method #1: Given coordinates of vertices & a triangulation:
Compute normals of triangles, average normals of triangles to estimate normal vectors at vertices.

When rendering - we will average (like smooth shading) normals from vertices to the get normals at pixels in the interior of the triangle. (Phong interpolation or Phong shading)
Method 2: Parametric Surfaces

\( \vec{f}(u, v) \) - defines a surface.

\( \vec{n}(u, v) \) - normal at the point \( \vec{f}(u, v) \).

Let \( \vec{n}(u, v) = \frac{\partial \vec{f}}{\partial v} \times \frac{\partial \vec{f}}{\partial u} = \vec{f}_u \times \vec{f}_v \)

or \( \vec{n}(u, v) = \frac{\vec{f}_u \times \vec{f}_v}{\| \vec{f}_u \times \vec{f}_v \|} \).
Example Paraboloid

\[ y = x^2 + z^2. \]

\[ x \text{ and } z \text{ are the parameters (i.e. instead of being called } u, v) \]

\[ \vec{f}(x, z) = \langle x, x^2 + z^2, z \rangle. \]

\[ f_x = \frac{\partial \vec{f}}{\partial x} = \langle 1, 2x, 0 \rangle \]

\[ \vec{f}_x \times \vec{f}_z = \langle 2x, -1, 2z \rangle \]

Answer is in terms of \( x, z \)

(i.e. in terms of \( u, v \)).

Be careful about the sign.

Alternative parameterization:

\[ r = \sqrt{x^2 + z^2}, \theta \text{-angle around } x \text{-axis} \]

\[ \theta = \arctan \left( \frac{z}{x} \right) \]

\[ f(r, \theta) = \langle r \sin \theta, r^2, r \cos \theta \rangle \]
Method #3: Level Set definition of a surface.

Let \( h(x,y,z) \) be a scalar-valued function. The surface \( S = \{ \langle x,y,z \rangle : h(x,y,z) = 0 \} \).

\[ \hat{n}(x,y,z) = \nabla h = \left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z} \right\rangle \]

- the gradient of \( h \).

Example Paraboloid: \( y = x^2 + z^2 \). Can be expressed \( \langle x,y,z \rangle : y - x^2 - z^2 = 0 \) \( h(x,y,z) = y - x^2 - z^2 \).

\[ \hat{n}(x,y,z) = \left\langle -2x, 1, -2z \right\rangle. \leftarrow \hat{n} \) is a function of \( x,y,z \) in general.

Watch out for sign.

In any method it may need to reversed.
Partial derivatives can be calculated when there is a mathematical function. Otherwise use a numerical approximation.
Example Ellipsoid \[ x^2 + 4y^2 + z^2 = 4 \]

Squashed sphere

\[ \nabla h = \langle 2x, 8y, 2z \rangle \quad \text{will normal to the ellipsoid at } \langle x, y, z \rangle \]

\[ \hat{n}(x, y, z) \text{ as a function of } x, y, z \]

given \( x, y, z \) is on the ellipsoid.
**Method 2:** Surface of rotation

Rotate $g$ around the $y$-axis, $g$ generates a surface of rotation.

**Parametric form:** $f(x,\theta) = \langle v \cdot \sin \theta, g(r), v \cdot \cos \theta \rangle$

**Level surface form:** $h(x, y, z) = \gamma - g(\sqrt{x^2 + z^2})$

**Shortcut method** Slope is $s = g'(r)$

A vector perpendicular (normal) is $\langle s, -1 \rangle$ (In $r$-$y$ plane)

or - better perhaps $\langle -s, 1 \rangle$

When rotate around $y$-axis, normals are $\langle -s \cdot \sin \theta, 1, -s \cdot \cos \theta \rangle$.
Transformations of normals:

Example: $\mathbb{R}^2$

\[ M = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \]

\[ S_{\left< \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right>} \]

\[ \left< \frac{\sqrt{2}}{2}, 1 \right> \text{ is a unit } \]

\[ \overline{v} \text{ is normal} \]

\[ \left< 2, 1 \right> \text{ is normal} \]

\[ M^{-1} \text{ does not yield } \left< 2, 1 \right> \]

\[ M \text{ does not correctly transform normal.} \]

Instead: \[ (M^{-1})^T = (M^T)^{-1} \] is used to transform normals.
**Theorem.** If \( \vec{n} \) is normal to surface \( \mathcal{S} \) at point \( \vec{x} \) and \( M \) is a 3x3 matrix for a linear transformation then \( (M^{-1})^T \vec{n} \) is normal to the surface \( M(\mathcal{S}) \) at \( M\vec{x} \).

(Same holds for \( M \) the linear part of affine transformations.)

**Proof:**

\[ \vec{n} \text{ is tangent to } \mathcal{S} \text{ at } \vec{x} \]

\[ \begin{align*}
\vec{n}^T (M^{-1})^T \vec{n} &= 0 \quad \text{if } \vec{n}^T \vec{t} = 0 \\
\vec{n}^T (M^{-1})^T (M \vec{t}) &= 0 \quad \text{if } \vec{n}^T \vec{t} = 0 \\
(\vec{n}^T (M^{-1})^T (M \vec{t}))^T &= 0 \\
\vec{n}^T (M^{-1})^T (M \vec{t}) &= \vec{n}^T (M^{-1})^T (M \vec{t}) = \vec{n}^T \vec{t} = 0
\end{align*} \]

So need to show: \( (M^{-1})^T \vec{n} \cdot M \vec{t} = 0 \) if \( \vec{n} \cdot \vec{t} = 0 \)

\[ \vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} \]

\[ (AD)^T = B^T A^T \]

\[ (A^T)^T = A \]

by assumption

QED
Phong lighting:
Point light source → •

$\hat{\ell}$, $\hat{v}$, $\hat{n}$ - unit vectors
$\hat{\ell}$ - light direction
$\hat{v}$ - normal vector
$\hat{n}$ - view direction

Surface

$\mathbf{r}$ - direction of perfect reflector

$\hat{\ell}$, $\hat{v}$, $\hat{n}$ - unit vectors

Ambient light - coming from all directions, reflects in all directions

Diffuse light - coming from a point light source, reflecting equally in all directions

Specular light - coming from a point light, reflecting more or less mirror-like