Last time I introduced the unit sphere 

Earth transformed by 

\[ \mathbf{R}_{\mathbf{E}} \circ T_{\langle 0,0,0 \rangle_{\mathbf{E}}} \circ \mathbf{R}_{\mathbf{E}} \circ \mathbf{S}_{0.7} \]

Earth is rotating on its axis and revolving around its axis, controlling the Earth-Moon system.

Moon transformed by

\[ \mathbf{R}_{\mathbf{E}} \circ T_{\langle 0,0,0 \rangle_{\mathbf{E}}} \circ \mathbf{R}_{\mathbf{M}} \circ T_{\langle 0,0,0 \rangle_{\mathbf{M}}} \circ \mathbf{S}_{0.4} \]
Linear Map R4's

Psuedo-code

$M_s, M_e, M_m, M_n$

$M_s = \text{Identity}$

$\text{Render } M_s (d) ; \text{Sun}$

$M_{E_0} = M_s$

$M_{E_0} = M_{E_0} \cdot R_{E_0, d}^>$

$M_{E_0} = M_{E_0} \cdot T_{0,0,0, v_E}$

$M_{E_1} = M_{E_0}$

$M_{E_1} = M_{E_1} \cdot R_{E_1, d}^>$

$M_{E_1} = M_{E_1} \cdot S_{0.7}$

$\text{Render } M_{E_1}(d) \text{ Earth}$

$M_m = M_{E_0}$

$M_m = M_m \cdot R_{E_0, m, j}$

$M_m = M_m \cdot T_{0,0, r_m}$

$M_s \text{ Set Identity ( );}$

$\text{Render the sun w/ } M_s$

$M_{E_0} = M_{E_0}$

$M_{E_0} \cdot \text{ Mult-g } \text{ Rotate } (0, 0, 0)$;

$M_{E_0} \cdot \text{ Mult-g } \text{ Translate } (0, 0, 0, v_E)$;

$M_{E_1} = M_{E_0};$

$M_{E_1} \cdot \text{ Mult-g } \text{ Rotate } (0, 0, 0)$;

$M_{E_1} \cdot \text{ Mult-g } \text{ Scale } (0.7)$;

$\text{Render the earth w/ } M_{E_1}$

$M_m = M_m \cdot S_{0.4}$

$\text{Render the } M_{m}(d) \text{ Moon}$
Back to rigid, orientation-preserving transformations in $\mathbb{R}^2$.

Recall $R_\theta$ - rotation angle $\theta$, CCW around $O$.

$R_\theta$ - linear, rigid, orientation preserving.

Generalized rotation in $\mathbb{R}^2$:

$R_\theta \vec{v} = \vec{v}$. - Affine.

Express $R_\theta$ as a composition of $R_\theta$'s, $T_{\vec{u}}$, $S_x$'s.

$R_\theta = T_{\vec{u}} \circ R_\theta \circ T_{-\vec{u}}$.
What are the possible rigid, orientation preserving maps \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \)?

**Theorem.** If \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) is ROP, and if \( A(\vec{u}) = \vec{v} \), then

\[ A \] is equal to \( R_{\theta} \) for some \( \theta \in \mathbb{R} \).

\[ A(\vec{u}) = \vec{v} \]

\[ \| A(\vec{u}) \| = 1 \quad \text{by rigidity} \]

\[ \| \vec{u} - \vec{v} \| = \| A(\vec{u}) - A(\vec{v}) \| \]

\[ \vec{u} \] - an arbitrary point in plane.

By rigidity, \( \| A(\vec{u}) \| \) and \( \| (A(\vec{u}) - A(\vec{v}) \| \) are equal (null and of \( \| \vec{u} - \vec{v} \| \)).

Only possibility for \( A(\vec{u}) \) is \( \vec{u} \) rotated by the same angle as \( A(\vec{u}) \) was rotated. Rigidity would allow \( -A(\vec{u}) \) as the value, but that is prohibited by orientation preserving...
Q.E.D.

Same argument shows:

Then if $A$ is an ROP, $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, and if $A(\mathbf{v}) = \mathbf{v}$, then $A = R_{\theta} \mathbf{v}$ for some $\theta$. 
Theorem If $A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a ROP, then either:

1. $A$ is a translation $T_{\vec{u}}$, or
2. $A$ is a generalized rotation $R_{\vec{v}}$.

Proof: Let $\vec{u} = A(\vec{0})$. Consider $A(\vec{u})$. If $\vec{u} = 0$, $A(\vec{0}) = 0$ so $A$ is a rotation.

- Case (1) $A(\vec{u}) = 2\vec{u}$, then $A$ is $T_{2\vec{v}}$.
- Case (2) $A(\vec{u}) = 0$.
- Case (3) Otherwise... we're in the situation of this picture.
In this case \( A \) is \( T \hat{u} \).

So \( A(\frac{1}{2} \hat{u}) = \frac{1}{2} \hat{u} \). So,

For \( \hat{v} = \frac{1}{2} \hat{u} \), \( A(\hat{v}) = \hat{v} \).

So \( A \cap R_{\theta} \).

\( \hat{v} \) - intersection of \( L_2, L'_2 \)

and \( A(\hat{v}) = \hat{v} \).

So \( A \cap R^v_{\theta} \) for some \( \theta \).
In $\mathbb{R}^3$

**Theorem:** If $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $A(\vec{0}) = \vec{0}$ and $A \parallel \text{ROP}$, then $A$ is a rotation $R_{\theta, \vec{u}}$ for some axis $\vec{u}$ (the axis is through the origin) and some $\theta \in \mathbb{R}$.

**Example:** Suppose $A(\vec{0}) = \vec{0}$, $A(\vec{i}) = \vec{j}$, $A(\vec{j}) = \vec{k}$, $A(\vec{k}) = \vec{i}$

i.e. $A(\langle x, y, z \rangle) = \langle z, x, y \rangle$.

Express $A$ in the form $R_{\theta, \vec{u}}$.

$\vec{u} = \langle 1, 1, 1 \rangle$, $\theta = 120^\circ$

$\vec{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

same axis, but a unit.
Theorem: Let $A: \mathbb{R}^2 \to \mathbb{R}^2$ be linear, with $2 \times 2$ matrix $M$.

Then $A$ is orientation preserving iff $\det(M) > 0$

Same holds in $\mathbb{R}^3$ (!)

$U'$ is $u$ rotated by $90^\circ$.

If $\vec{u} = (u_1, u_2)$, then $\vec{u}' = (-u_2, u_1)$

We have: $\vec{v}$ is "to the left of $\vec{u}$" iff $\vec{u}' \cdot \vec{v} > 0$

$\vec{u}' \cdot \vec{v}$ is

\[-u_2 v_1 + u_1 v_2 = \det \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}\]