Moving to $\mathbb{R}^3$ - (3-space)

$$\vec{x} = \langle x_1, x_2, x_3 \rangle = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$z$-axis towards the viewer.

$$\vec{x} \times \vec{y} = \vec{k} \quad \text{obeys right hand rule for cross product}$$

$\vec{u} \times \vec{v}$

Defin Linear transform - same definition as before.

Affine transformation - 

$$A(\vec{x}) = B(\vec{x}) + \vec{u} \quad \text{where } B \text{ is linear}$$
Translation $T_{\vec{u}}$ for $\vec{u} \in \mathbb{R}^3$

$T_{\vec{u}}(\vec{x}) = \vec{x} + \vec{u}$.

Example $\vec{u} = <1, 0, 0>$. $T_{\vec{u}}(<x, y, z>) = <x + 1, y, z>$

$3 \times 3$ matrix representation of a linear map $A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$

If $A(\vec{u}) = \vec{u}$, $A(\vec{j}) = \vec{v}$, $A(\vec{k}) = \vec{w}$

then $A$ is represented by

\[
\begin{pmatrix}
\vec{u} & \vec{v} & \vec{w}
\end{pmatrix}
\begin{pmatrix}
\vec{u}_1 & \vec{v}_1 & \vec{w}_1 \\
\vec{u}_2 & \vec{v}_2 & \vec{w}_2 \\
\vec{u}_3 & \vec{v}_3 & \vec{w}_3
\end{pmatrix}
\]

e.g. $M \vec{u} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \end{pmatrix} = \vec{u}$. 
Example: The rotation $R_{90^\circ,\hat{j}}$ or $R_{\frac{\pi}{2},\hat{j}}$ rotates around the vector $\hat{j}$ (y-axis) $90^\circ$ in the counterclockwise direction viewed from above (as given by the right-hand rule).

Question: What 3×3 matrix represents $R_{\frac{\pi}{2},\hat{j}}$?

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0 \\
\end{pmatrix}
\]

Acting on homogeneous coordinates use:

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Homogeneous coordinates

Refresh in $\mathbb{R}^2$: $<x, y, w>$ homogeneous coordinates for $<x/w, y/w>$

$$A(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x + \begin{pmatrix} e \\ f \end{pmatrix}$$ - affine

The $3 \times 3$ matrix

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}$$

$<3, 5, 1>, <3/2, 5/2, 1/2>, <6, 10, 2>$

all represent $<3, 5> \in \mathbb{R}^2$
In $\mathbb{R}^3$, $<x, y, z, w>$ homogeneous coordinates for $<x', y', z'>$

An affine map $A: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$A\vec{x} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix} \vec{x} + \begin{pmatrix} m \\ n \\ p \end{pmatrix}$$

represent by

$$\begin{pmatrix} a & b & c & m \\ d & e & f & n \\ g & h & l & p \\ 0 & 0 & 0 & 1 \end{pmatrix} = M$$

$M \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ gives $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ where $A\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
Example

Uniform Scaling

\[ S_\alpha (\langle x, y, z \rangle) = \langle \alpha x, \alpha y, \alpha z \rangle \]

Nonuniform scaling

\[ S_{\langle \alpha, \beta, \gamma \rangle} (\langle x, y, z \rangle) = \langle \alpha x, \beta y, \gamma z \rangle \]

These are linear.

\[ S_{\langle \alpha, \beta, \gamma \rangle} \text{ represented 3x3 matrix} \]

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma \\
\end{pmatrix}
\]

\[ S_{\alpha \beta \gamma} (\vec{g}) = (\vec{g})^\prime \]
A transformation $A$ is rigid if for all $\vec{x}, \vec{y}$, 

$$\|A(\vec{x}) - A(\vec{y})\| = \|\vec{x} - \vec{y}\| \quad \text{ i.e.,}$$

$A$ preserves distances (between points).

Example: A translation $T_u$ or a rotation $R_{\vec{u}}$ are rigid.

"Rigid" means sizes or shapes do not change (except...!!)
Also a reflecting example $S_{<1,1,1>}$

$$A(\langle x, y, z \rangle) = \langle -x, y, z \rangle$$

is also rigid. (So "except..." refers to reflections).

Observation: A rigid transformation also preserves angles.

By SSS theorem (Side-Side-Side)
Def: In $\mathbb{R}^2$, $A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is orientation preserving if it preserves directions of angles.

Counterwise -ness of angle is unchanged.

So is orientation preserving.

Not orientation preserving.
In $\mathbb{R}^3$, a transformation $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is orientation preserving if it preserves the sign of triple products $(\vec{u} \times \vec{v}) \cdot \vec{w}$.

Iff: $(A(\vec{u}) \times A(\vec{v})) \cdot A(\vec{w}) > 0$.

I.e., intuition that right-handness of orientations of triples of vectors

If $\vec{u} = A(\vec{u})$, $\vec{v} = A(\vec{v})$, $\vec{w} = A(\vec{w})$.

Plane with $\vec{u} \times \vec{v}$. $\vec{w}$ "above" $\vec{u} \in \vec{v}$.
In $\mathbb{R}^3$, Translations $T_v$

Rotations $R_{\theta, \vec{u}}$

Scaling $S_{\alpha, \beta, r}$ with $\alpha, \beta, r > 0$

are all orientation preserving
Suppose we are modeling a solar system: Sun, Earth, Moon. Use a "top view" (looking down the x-axis).

\( \theta_E \) - angle Earth is revolved around the Sun.

\( \theta_m \) - angle Moon has revolved around Earth.

\( r_E, r_m \) - radii of the orbits.

\( \varphi \) - how much Earth is rotated on its axis.

This picture shows Earth moving around the Sun. The unit sphere at \( \varphi \) will render Sun Earth & Moon.

Volumetric transformations
1. Render Sun as just $f$.
2. Render Earth as a scaled, rotated, translated version of $f$.

Let $M_1 = R_{\phi_E,j} \circ T_{c_0,0,v_E} \circ R_{\phi,j} \circ S_{0.7}$. Render Earth as $M_1 f$.

$$M_1 = \begin{pmatrix} \cos \phi & -\sin \phi & c_0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & v_E \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_{0.7} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
For moon Use:

\[ M_{\text{moon}} = R_{E,j} \circ T_{C,0,v_E} \circ R_{m,j} \circ T_{C,0,v_m} \circ S_{0,0} \]

Render man as \( M_m (\theta) \)

Last two operations

Let the Earth transform also affect the moon also.

So \( R_{E,j}, T_{C,0,v_E} \) affect the whole Earth system.