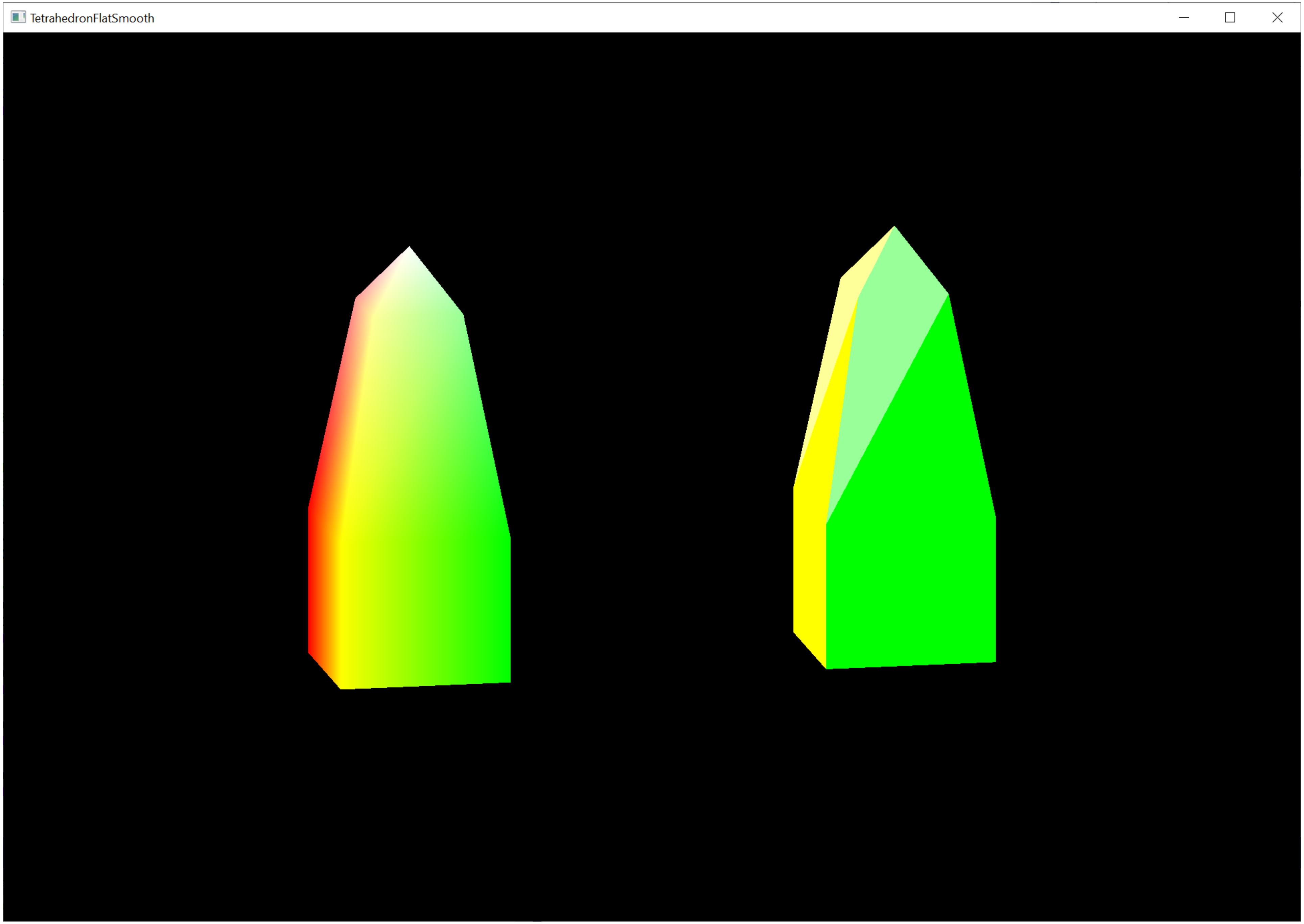
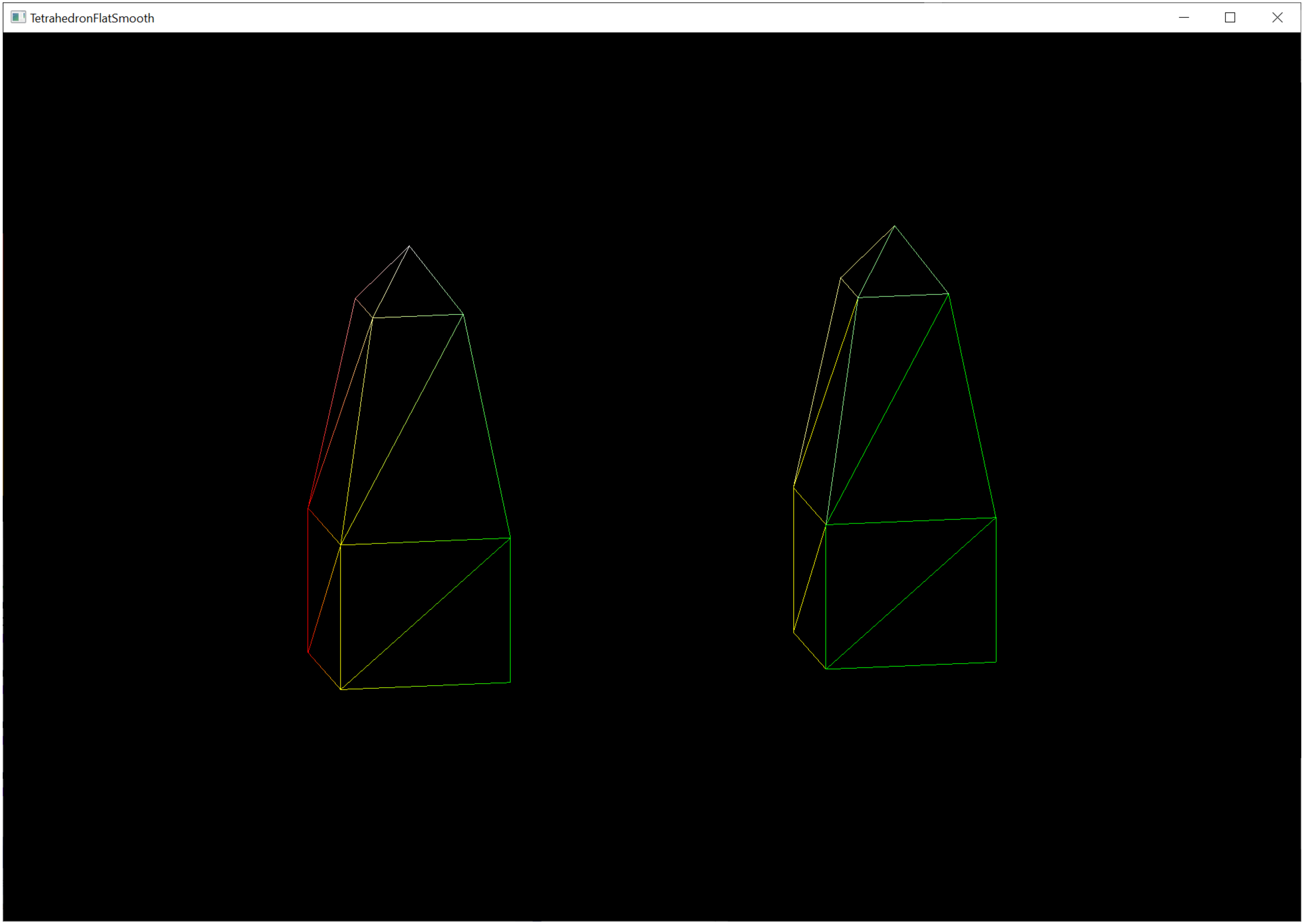
Math 155A - Lecture 3 - September 29


TetrahedronFlatSmooth


TetrahedronFlatSmooth

Animation: Successive fixed images
- create appearance of motion.

Front buffer – Currently being displayed

Back buffer – Holds the next image as it is being formed.
- Not visible!

Swap buffers to display the next image.

Linear transformations: (In $\mathbb{R}^2$)

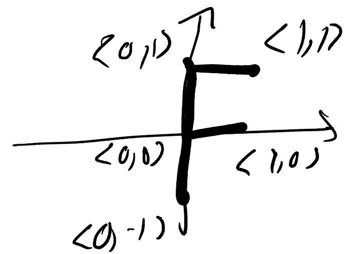
Examples:

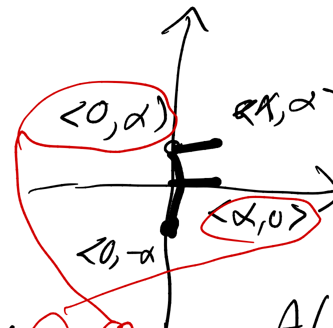
Id $\qquad Id(\vec{x}) = \vec{x} \qquad$ Matrix: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Uniform scaling

$$S_{1/2}(\vec{x}) = \tfrac{1}{2}\vec{x}$$

$$S_\alpha(\vec{x}) = \alpha\vec{x}$$





Matrix representation of S_α is $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$

$A(\vec{i}) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

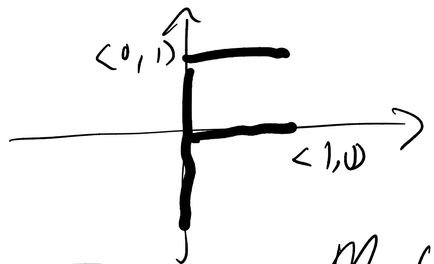
$A(\vec{j}) = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$

Non-uniform scaling

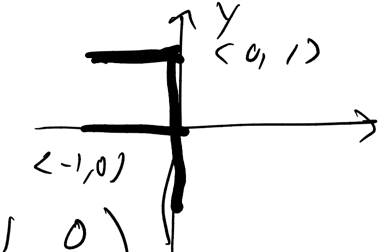
$$S_{\alpha,\beta} = S_{\langle \alpha,\beta\rangle} \qquad S_{\alpha,\beta}(\langle x,y\rangle) = \langle \alpha x, \beta y\rangle$$

F $\Rightarrow$ 

Reflection across y-axis



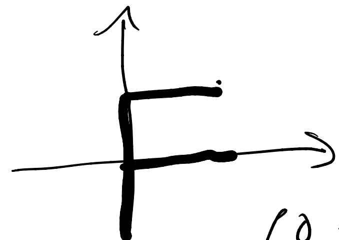
$\Rightarrow$



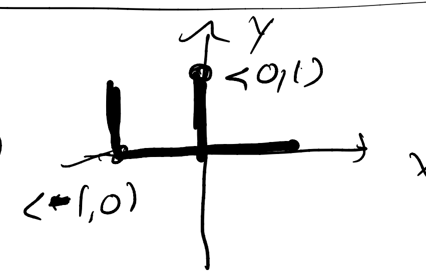
Matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Rotation by 90° counterclockwise

$R_{90°}$ or $R_{\pi/2}$



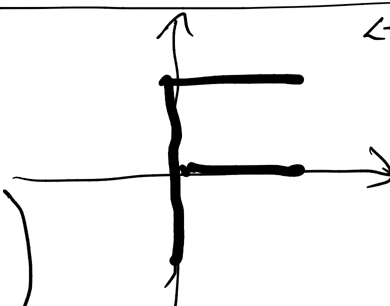
$\Rightarrow$



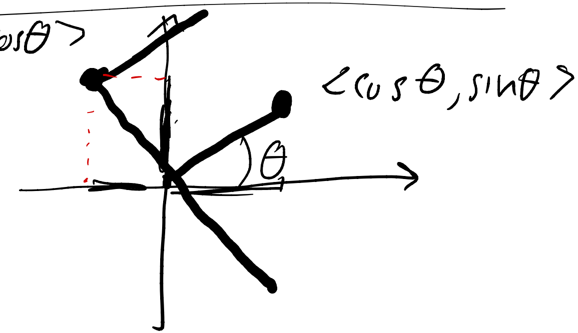
Matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Rotation by angle θ counterclockwise (around the origin)

Matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$\Rightarrow$



Composition: Let $A, B: \mathbb{R}^2 \to \mathbb{R}^2$

The composition $A \circ B$ is the map s.t

$$(A \circ B)(\vec{x}) = A(B(\vec{x}))$$

i.e. Apply B to $\vec{x}$, then apply A to the result.

Eg $\quad Id \circ A = A$

Inverse: Let $A: \mathbb{R}^2 \to \mathbb{R}^2$. The inverse of A (if any) is the unique A^{-1} s.t

$$A^{-1} \circ A = Id \quad \text{and} \quad A \circ A^{-1} = Id.$$

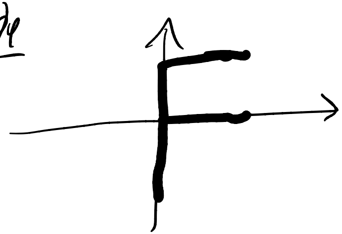
Affine transformations:

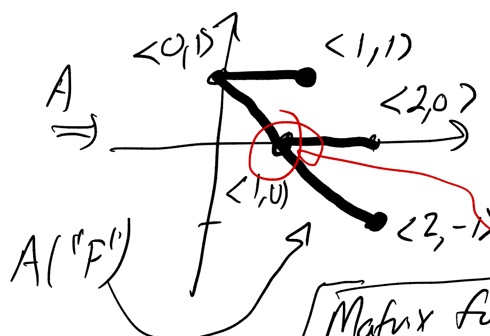
An affine transformation (of $\mathbb{R}^2$) is a mapping A of the form

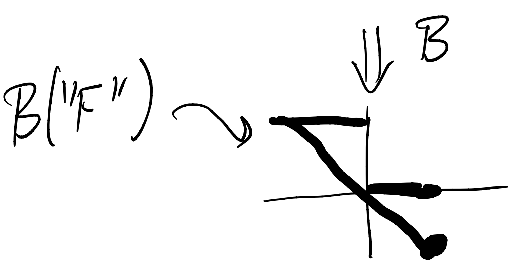
$$A(\vec{x}) = B(\vec{x}) + \vec{u} \quad \text{where}$$

- B is linear, and
- $\vec{u} \in \mathbb{R}^2 \leftarrow$ translation (or displacement)

Example







$$A(\vec{x}) = B(\vec{x}) + \vec{u}$$

where $u = \langle 1, 0 \rangle$

Matrix for B: $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$A(\vec{x}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- If A is linear, $A(\vec{0}) = \vec{0}$. (An affine transformation with $u = 0$.)

<u>Thm</u> • If A is affine, there is a unique linear map B and $u \in \mathbb{R}^2$ such that $A(\vec{x}) = B(\vec{x}) + \vec{u}$

<u>Pf</u>: $\vec{u} = A(\vec{0})$ since $A(\vec{0}) = B(\vec{0}) + \vec{u} = \vec{0} + \vec{u} = \vec{u}$

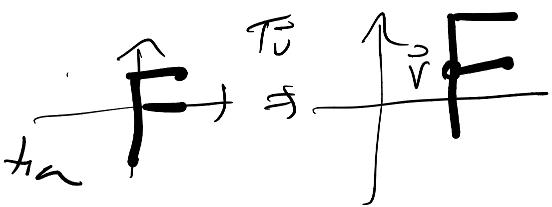
and $\boxed{B = T_{-u} \circ A}$ ← Since $A(\vec{x}) = B(\vec{x}) + \vec{u}$,

$$B(\vec{x}) = A(\vec{x}) + (-\vec{u})$$
$$= (T_{-u} \circ A)\,\vec{x}$$

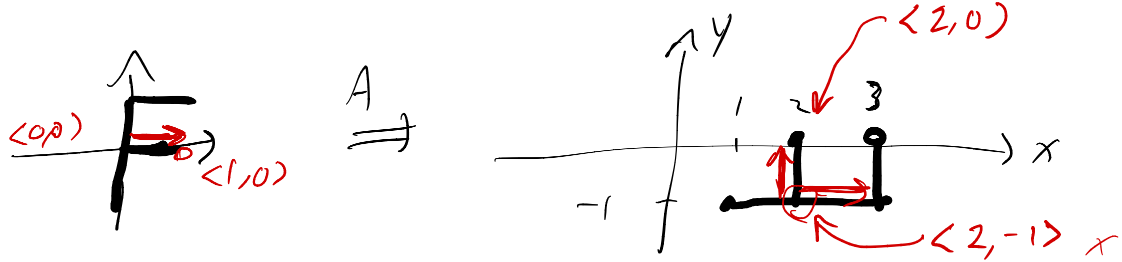
<u>Notation</u>: $T_{\vec{v}}(\vec{x}) = \vec{x} + \vec{v}$

$T_{\vec{v}}$ is an affine transformation

"translation by $\vec{v}$"



Example



Express A in form $A(\vec{x}) = N\vec{x} + \vec{u}$

where N is a 2×2 matrix and $\vec{u} \in \mathbb{R}^2$

$$A(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\underbrace{\langle 2,0\rangle}_{A(\vec{\imath})} - \underbrace{\langle 2,-1\rangle}_{A(\vec{0})} = \langle 0,1\rangle$$

$$(B(\vec{\imath}) + \vec{u}) - (B(\vec{0}) + \vec{u}) = B(\vec{\imath})$$

Matrices for inverses

Suppose $A(\vec{x})$ is linear, how to find matrix for A^{-1} ?

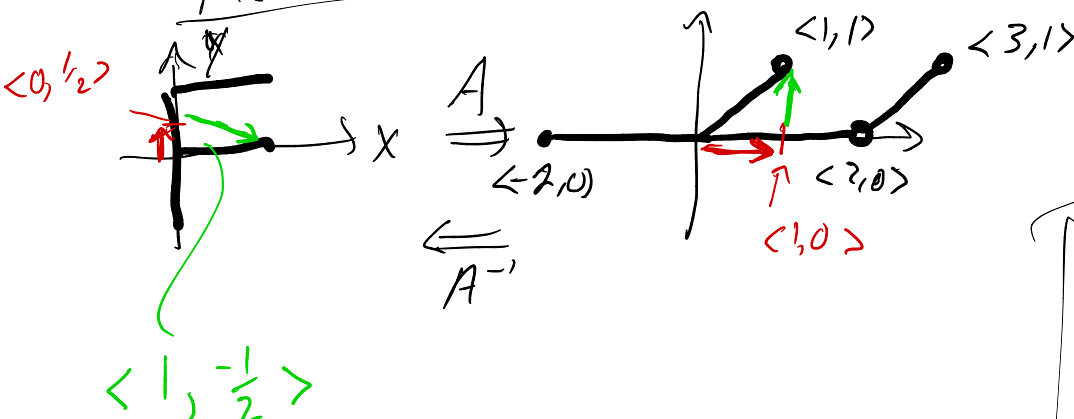
Method #1 If $A(\vec{x}) = M\vec{x}$ M - 2x2 matrix

Then M^{-1} represents A^{-1}

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & d \end{pmatrix} \frac{1}{ad-bc}$$

ad-bc
-determinant

Method #2 Visualize from the "F"



Matrix for A^{-1}: $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

Matrix for A is $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Easy to check by multiplying

For inverses of affine maps

Let $A(\vec{x}) = B(\vec{x}) + \vec{u}$ be affine

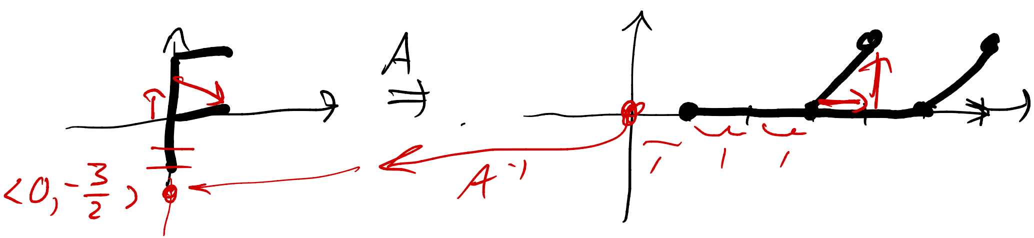
Method #1 Find A^{-1} as follows,

Let $\vec{y} = B(\vec{x}) + \vec{u}$ & solve for $\vec{x}$ in terms of y

$$B(\vec{x}) = \vec{y} - \vec{u}$$

$$\vec{x} = B^{-1}(\vec{y} - \vec{u}) = B^{-1}(\vec{y}) - B^{-1}(\vec{u}) = B^{-1}(\vec{y}) + (-B^{-1}(\vec{u}))$$

Method #2 - Visual method



Linear of this A is same as the linear part of the previous A. Also, the linear part of A^{-1} is the same as befor

$$A^{-1}(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ -3/2 \end{pmatrix}.$$