1. [20 points] (Linear interpolation/extrapolation) Let $\mathbf{x} = \langle 2, 0\rangle$ and $\mathbf{y} = \langle -4, 1\rangle$.

a. What is $\mathbf{u} = \text{LERP}(\mathbf{x}, \mathbf{y}, \frac{1}{4})$? (Give $\mathbf{u}$ explicitly.)

b. What is $\mathbf{v} = \text{LERP}(\mathbf{x}, \mathbf{y}, 2)$. (Give $\mathbf{v}$ explicitly.)

c. Suppose f is a function with $f(\mathbf{u}) = 0$ and $f(\mathbf{v}) = 10$, and we want set f's values for other points by linear interpolation or extrapolation. What will this give for the value of $f(0, \frac{1}{3})$?

d. What point $\mathbf{r}$ on the line containing $\mathbf{x}$ and $\mathbf{y}$ is closest to the origin $\mathbf{0}$? Express your answer in the form $\mathbf{r} = \text{LERP}(\mathbf{x}, \mathbf{y}, \alpha)$ for some value of α.

2. [20 points] This problem concerns interpolation in homogeneous coordinates or hyperbolic interpolation. Suppose that two 4-vectors $\mathbf{v}$ and $\mathbf{w}$ are equal to

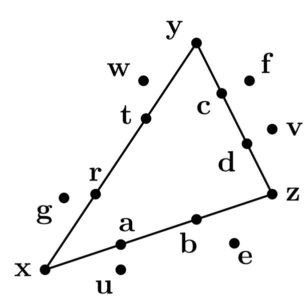
$$\mathbf{v} = \langle 4, 0, 4, 2\rangle \qquad \text{and} \qquad \mathbf{w} = \langle 0, 0, 0, 1\rangle.$$

These are homogeneous coordinates for the points $\langle 2, 0, 2\rangle$ and $\langle 0, 0, 0\rangle$ in $\mathbb{R}^3$.

2.a. Find a value for α so that $\text{LERP}(\mathbf{v}, \mathbf{w}, \alpha)$ is a homogeneous representation of the midpoint $\langle 1, 0, 1\rangle$ in $\mathbb{R}^3$.

2.b. Write out $\text{LERP}(\mathbf{v}, \mathbf{w}, \alpha)$ explicitly (as a 4-vector).

3. [20 points] Refer to the figure below, for questions about barycentric coordinates $\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z}$. For part b., let $\mathbf{x} = \langle -2, -1\rangle$ and $\mathbf{y} = \langle 0, 1\rangle$ and $\mathbf{z} = \langle 1, 0\rangle$. (Not drawn to scale.)



a. Answer the following questions with the appropriate answer chosen from **a** through **z**:

i. Which point has barycentric coordinates $\alpha=\frac{1}{3}$, $\beta=0$, $\gamma=\frac{2}{3}$?

ii. Which point has barycentric coordinates $\alpha=\frac{3}{7}$, $\beta=\frac{5}{7}$, $\gamma=\frac{-1}{7}$?

b. What are the barycentric coordinates for the point $\mathbf{h} = \mathbf{0} = \langle 0, 0\rangle$? Draw its approximate location on the figure above.

4. [20 points] Briefly describe the three listed methods below. Make it clear how they differ.

(a) Supersampling,

(b) Stochastic Supersampling, and

(c) Jittered Stochastic Supersampling.

5. [10 points] A parabolic surface S is defined as the set of points $\langle x, y, z\rangle \in \mathbb{R}^3$ which satisfy $y = -(x^2 + \frac{1}{2}z^2)$. For a general point $\mathbf{u} = \langle x, y, z\rangle$ on S, give the formula for a vector normal to the surface at $\mathbf{u}$. Your vector does not need to be a normal vector, but it should be pointing upward from the surface, not downward.

6. [20 points] Give the formula for the diffuse component of Phong lighting. Also, explain what all the variables represent and any special properties they should satisfy. (For example, do they need to be unit vectors?) Third, draw a picture showing the surface vertex, the light and eye positions, and any relevant vectors. (You should do this for a single color and single light: you do **not** need to describe how to handle multiple lights and multiple colors.)

5. [10 points] A hyperboloid surface $\mathcal{S}$ is defined as

$$\mathcal{S} \;=\; \{\langle x, y, z\rangle : y^2 = 10 + x^2 + 2z^2\}.$$

For $\mathbf{u} = \langle x, y, z\rangle$ a point on $\mathcal{S}$, give an equation for a normal vector $\mathbf{n}$, i.e., normal to $\mathcal{S}$ at the point $\mathbf{u}$. Your vector does not need to a unit vector, but it should be pointing ***towards*** the xz-plane.

1. Let the flattened ellipsoid E have radii 2, 1, 2, and be defined by the equation $x^2 + 4y^2 + z^2 = 4$. Where does the ellipsoid intersect the three axes?

a. Draw a picture by hand, illustrating E as best you can.

b. Suppose $\langle x, y, z\rangle$ is a point on the ellipse. Give a formula for the outward normal at the ellipse at this point.

b. Give a parametric equation for E; that is, a function f of two parameters (for instance, θ and ϕ) so that $f(\theta, \phi)$ gives the points on the ellipsoid as its two parameters vary. What ranges do the parameters need to vary over? *Hint: do this similarly to the parametric description of a sphere in spherical coordinates.*

c. Use the parametric equation for E to give a formula for the outward unit normal at a point on E. Your formula for the normal should be in terms of θ and ϕ.

2. A cone C has tip at $\langle 0, 1, 0\rangle$, and its base is the disk $x^2 + z^2 \leq 1$ in the xz-plane. Suppose $\langle x, y, z\rangle$ is a point on the side of the ellipse (not on the base). Give a formula for the outward unit normal to the cone at this point.

3. Let $\mathbf{x} = \langle -2, 2\rangle$ and $\mathbf{y} = \langle 2, 0\rangle$. Let α control linear interpolation and extrapolation from x to y. What five points are obtained when $\alpha = -2$, $\alpha = 0$, $\alpha = \frac{1}{2}$, $\alpha = 1$, and $\alpha = 2$? What value of α gives the point $\langle 1.6, 0.2\rangle$? Graph by hand all these values, labeling things clearly.

4. Continuing problem 3. If you apply the formula for inverting linear interpolation to the origin $\langle 0, 0\rangle$, what value of α do you get? Use this to compute the point on the line containing $\mathbf{x}$ and $\mathbf{y}$ that is closest to the origin.

5. Let $\mathbf{x} = \langle -1, -2\rangle$, and $\mathbf{y} = \langle 1, 1\rangle$, and $\mathbf{z} = \langle 1, -1\rangle$. What points are obtained by the following sets of barycentric coordinates?

a. $\alpha = 0$, $\beta = 0$, and $\gamma = 1$.

b. $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$, and $\gamma = 0$.

c. $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$, and $\gamma = \frac{1}{3}$.

d. $\alpha = \frac{4}{5}$, $\beta = \frac{1}{10}$, and $\gamma = \frac{1}{10}$.

e. $\alpha = \frac{4}{3}$, $\beta = \frac{2}{3}$, and $\gamma = -1$.

6. For the same triangle as in the previous problem, what are the barycentric coordinates of the following points?

a. $\mathbf{u} = \langle 1, 1\rangle$.

b. $\mathbf{u} = \langle \frac{1}{3}, 0\rangle$.

c. $\mathbf{u} = \langle \frac{1}{2}, -\frac{1}{2}\rangle$.

7. For the same triangle again: Draw a graph showing where the points lie that have barycentric coordinates with $\alpha > 0$, $\beta > 0$, and $\gamma < 0$.

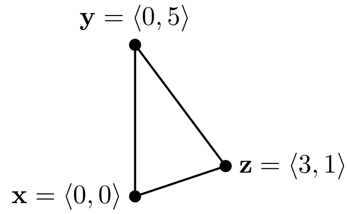
8. Let $\mathbf{x} = \langle 0, 0, 0\rangle$, $\mathbf{y} = \langle 5, 0, 1\rangle$, $\mathbf{z} = \langle 4, 1, 1\rangle$, and $\mathbf{w} = \langle -1, 2, 0\rangle$ be the four vertices of a quadrangle in counterclock-wise order. For each pair of values α and β, what point is obtained by bilinear interpolation in this quadrangle? (Or, if no such point exists, explain why not.)

a. $\alpha = 0$ and $\beta = 1$.

b. $\alpha = 1$ and $\beta = 1$.

c. $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$.

d. $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{3}$.



5. [20 points] Consider the triangle lying in $\mathbb{R}^2$ with vertices $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \langle 0, 5\rangle$, and $\mathbf{z} = \langle 3, 1\rangle$.

(a) What point in $\mathbb{R}^2$ has barycentric coordinates $\alpha = \frac{1}{2}$, $\beta = \frac{1}{6}$, and $\gamma = \frac{1}{3}$ relative to this triangle?

(b) What are the barycentric coordinates for the points $\langle 3, 1\rangle$? For the point $\langle 1, 3\rangle$?

6. [20 points] A patch $\mathbf{f}(\alpha, \beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 0, 0, 0\rangle$, $\mathbf{y} = \langle 6, 0, 3\rangle$, $\mathbf{z} = \langle 6, 6, 0\rangle$, and $\mathbf{w} = \langle 0, 6, 0\rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$.

(a) Give the parametric formula $\mathbf{q}(u, v)$ for the patch.

(b) What is the point on this patch with bilinear coordinates $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$?

(c) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

4. [20 points]

(a) What does the term "shading" mean in computer graphics?

(b) Briefly describe why shading is important for rendering 3D graphics images.

(c) Briefly describe Gouraud shading and Phong shading and how they are different.

(d) Compare these two kinds of shading. What are the relative advantages and disadvantages of Phong and Gouraud shading?

5. [20 points] Consider the triangle lying in $\mathbb{R}^2$ with vertices $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \langle 1, 3\rangle$, and $\mathbf{z} = \langle 5, 0\rangle$.

(a) What point in $\mathbb{R}^2$ has barycentric coordinates $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, and $\gamma = \frac{1}{6}$ relative to this triangle?

(b) What are the barycentric coordinates of the point $\langle 3, 1\rangle$?

6. [20 points] A patch $\mathbf{f}(\alpha, \beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 0, 0, 0 \rangle$, $\mathbf{y} = \langle 4, 0, 1 \rangle$, $\mathbf{z} = \langle 4, 4, 0 \rangle$, and $\mathbf{w} = \langle 0, 4, 0 \rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$.

(a) What is the point on this patch with bilinear coordinates $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$?

(b) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

3. (20 points) Barycentric coordinates. Let a triangle in the xy-plane be formed by the three points $\vec{x} = (0, 0)$, $\vec{y} = (3, 0)$ and $\vec{z} = (1, 3)$.

a. Explain (one or two sentences) the definitions of barycentric coordinates for points $\vec{u}$ with respect to this triangle.

b. What are the barycentric coordinates for the point $(1, 3)$?

c. What are the barycentric coordinates for the point $(2, 0)$?

4. (30 points) Give an description of how **specular** reflection is modeled in the Phong reflection model. Include a description of the various vectors and angles and their properties. Include the treatment of multiple light sources. **Include formulas** and explain the terms in the formulas. Explain common shortcuts used in the calculations. You should not include OpenGL specific enhancements such as attenuation and spotlights.

It is best if your description is both complete and succinct. You do not need to explain things at great length, but your answer should make it clear you understand how specular reflection is modeled.

1. [20 points] (Linear Interpolation/Extrapolation.) Let $\mathbf{x} = \langle 1, 2, 0 \rangle$ and $\mathbf{y} = \langle 0, 6, 2 \rangle$. Let L be the line containing $\mathbf{x}$ and $\mathbf{y}$.

a. What is $\textsc{Lerp}(\mathbf{x}, \mathbf{y}, \frac{1}{3})$?

b. What is $\textsc{Lerp}(\mathbf{x}, \mathbf{y}, 1)$?

c. Give the value of α such that $\mathbf{u} = \textsc{Lerp}(\mathbf{x}, \mathbf{y}, \alpha)$ is the point on L which is closest to the origin $\langle 0, 0, 0 \rangle$.

1. [20 points] (Linear interpolation) Let $\mathbf{x} = \langle -2, 0, 0\rangle$ and $\mathbf{y} = \langle 2, 1, 2\rangle$.

a. What is $\mathbf{u} = \textsc{Lerp}(\mathbf{x}, \mathbf{y}, \frac{1}{4})$? (Give $\mathbf{u}$ explicitly.)

b. What is $\mathbf{v} = \textsc{Lerp}(\mathbf{x}, \mathbf{y}, 1)$. (Give $\mathbf{v}$ explicitly.)

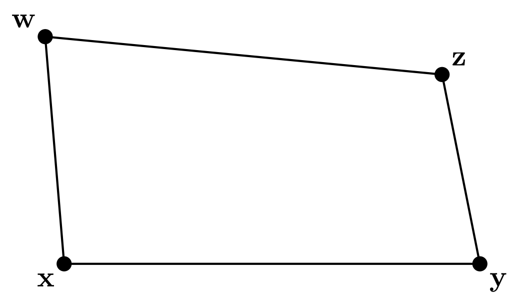
c. Let $\mathbf{z} = \langle 1, \frac{3}{4}, \frac{3}{2}\rangle$. What value α satisfies $\mathbf{z} = \textsc{Lerp}(\mathbf{x}, \mathbf{y}, \alpha)$?

d. Let $\mathbf{w} = \langle -2, 1, 0\rangle$. What point $\mathbf{r}$ on the line containing $\mathbf{x}$ and $\mathbf{y}$ is closest to $\mathbf{w}$?

2. [20 points] This problem concerns bilinear interpolation

$$\mathbf{u}(\alpha, \beta) = (1-\alpha)(1-\beta)\mathbf{x} + \alpha(1-\beta)\mathbf{y} + \alpha\beta\mathbf{z} + (1-\alpha)\beta\mathbf{w}$$

in $\mathbb{R}^3$. Let $\mathbf{x} = \langle -1, -1, 0\rangle$ and $\mathbf{y} = \langle 1, -1, 2\rangle$ and $\mathbf{z} = \langle 1, 1, 1\rangle$ and $\mathbf{w} = \langle -1, 1, 1\rangle$.



a. On the picture above draw the approximate locations of the following points and label them. (You do not need to calculate the points, just draw their approximate location.)

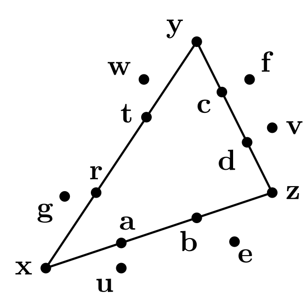
The point $\mathbf{s}$ with bilinear coordinates $\alpha = 0.9$ and $\beta = 0.1$.

The point $\mathbf{t}$ with bilinear coordinates $\alpha = 0$ and $\beta = 1$.

The point $\mathbf{u}$ with bilinear coordinates $\alpha = 0.5$ and $\beta = 0.5$.

b. The bilinear interpolation on $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{w}$ defines a (parametric) surface $\mathbf{q}(\alpha, \beta)$. Give a normal vector for this surface at the point $\mathbf{u}$ given above.

3. [20 points] Refer to the figure below, for questions about barycentric coordinates $\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z}$. For part b., let $\mathbf{x} = \langle -4, -2\rangle$ and $\mathbf{y} = \langle -2, 1\rangle$ and $\mathbf{z} = \langle 0, 0\rangle$.



a. Answer the following questions with the appropriate answer chosen from **a** through **z**:

i. Which point has barycentric coordinates α=0, β=$\frac{2}{3}$, γ=$\frac{1}{3}$?

ii. Which point has barycentric coordinates α=$\frac{5}{7}$, β=$\frac{-1}{7}$, γ=$\frac{3}{7}$?

b. What are the barycentric coordinates for the point $\mathbf{h} = \langle -1, 0\rangle$? Draw its approximate location on the figure above.

4. [15 points]

a. Give the definition of an "Affine Transformation" mapping $\mathbb{R}^n \to \mathbb{R}^n$.

b. Give the definition of an "Affine Combination" of k points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbb{R}^n$.

5. [20 points] Give the formula for the specular component of Phong lighting. Also, explain what all the variables represent and any special properties they should satisfy. (For example, do they need to be unit vectors?) Third, draw a picture showing the surface, the light and the eye position, and the relevant vectors. You do not need to discuss the "halfway vector", just the main Phong formula for specular lighting.

6. [20 points] A smooth surface S in $\mathbb{R}^3$ is transformed by the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ represented by the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 4 & 0 \end{pmatrix}$$

to form the surface $f(S)$. Give a 3×3 matrix B such that whenever $\mathbf{n}$ is normal to S at a point $\mathbf{x}$ on S, then the vector $\mathbf{m} = B\mathbf{n}$ is normal to the point $f(\mathbf{x})$ on the surface $f(S)$. [Hint: It is not difficult to invert A.]

10. [20 points] A patch $\mathbf{p}(\alpha, \beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 1, 0, 1 \rangle$, $\mathbf{y} = \langle 1, 2, -1 \rangle$, $\mathbf{z} = \langle -1, 2, 1 \rangle$, and $\mathbf{w} = \langle -1, 0, -1 \rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$. (You may wish to work this problem on scratch paper first, and then transfer your work to the exam.)

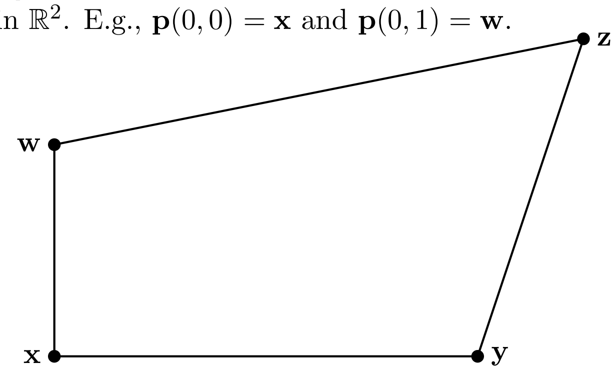
(a) The point $\mathbf{v} = \langle -1, \frac{3}{2}, \frac{1}{2} \rangle$ lies on the line segment joining $\mathbf{z}$ and $\mathbf{w}$. What are the bilinear coordinates α and β for the point $\mathbf{v}$?

(b) What is the point $\mathbf{u}$ on this patch with bilinear coordinates $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$?

(c) Give the values of the partial derivatives at this point $\mathbf{u}$: $\frac{\partial \mathbf{p}}{\partial \alpha}(\frac{1}{4}, \frac{1}{2})$ and $\frac{\partial \mathbf{p}}{\partial \beta}(\frac{1}{4}, \frac{1}{2})$.

(d) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

3. [20 points] Bilinear interpolation is used to define a surface $\mathbf{p}(\alpha, \beta)$ from four points $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ in $\mathbb{R}^2$. E.g., $\mathbf{p}(0,0) = \mathbf{x}$ and $\mathbf{p}(0,1) = \mathbf{w}$.



a. Draw and label on the figure above the approximate locations of the points:

- $\mathbf{a} = \mathbf{p}(0.1, 0.9)$,
- $\mathbf{b} = \mathbf{p}(\frac{1}{2}, \frac{1}{2})$,
- $\mathbf{c} = \mathbf{p}(\frac{1}{3}, 1)$.

b. Define $\mathbf{q}(\alpha) = \mathbf{p}(\alpha, 1)$. What is the formula for the first derivative $\mathbf{q}'(\alpha)$? (For parts (b) and (c), express your answer in terms of $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$.)

c. Define $\mathbf{r}(\alpha) = \mathbf{p}(\alpha, \alpha)$. What is the formula for the first derivative $\mathbf{r}'(\alpha)$?