Kantorovich Duality & Optimal Transport Problems on Magnetic Graphs

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Magnetic Graphs & Lifts

A combinatorial graph $G = (V(G), E(G))$ is called **simple** if its vertex set is finite and its edge set contains no loops or multiple edges. A graph is called **connected** if there is at least one path connecting any two vertices. Throughout, we consider simple, connected graphs. If two vertices $u, v \in V$ are adjacent, we write $u \sim v$.

Signatures

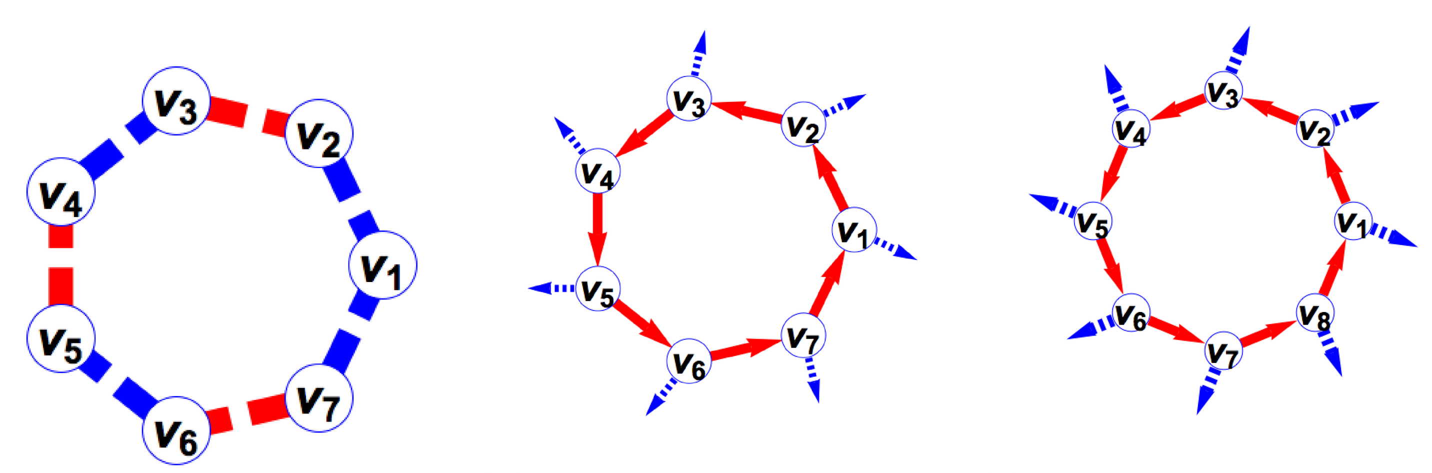
The **oriented edge set** of a graph G is given by

$$E^{or}(G) := \{(u,v),(v,u) : u,v \in V(G), u \sim v\}.$$

A **signature** on a graph is a map

$$\sigma : E^{or}(G) \to \mathbf{S}^1 : (u,v) \mapsto \sigma_{uv},$$

satisfying the property $\sigma_{vu} = \overline{\sigma_{uv}}$. A pair (G,$\sigma$)*iscalleda***magnetic graph**.



(a) 7-vertex cycle graph, with real-valued signature. The edges with positive signature are in blue, those with negative signature are in red.

(b) 7-vertex cycle graph with complex-valued signature. All edges have signature $e^{\frac{i\pi}{2}}$, illustrated by the angular offset of the blue arrow from the red edges.

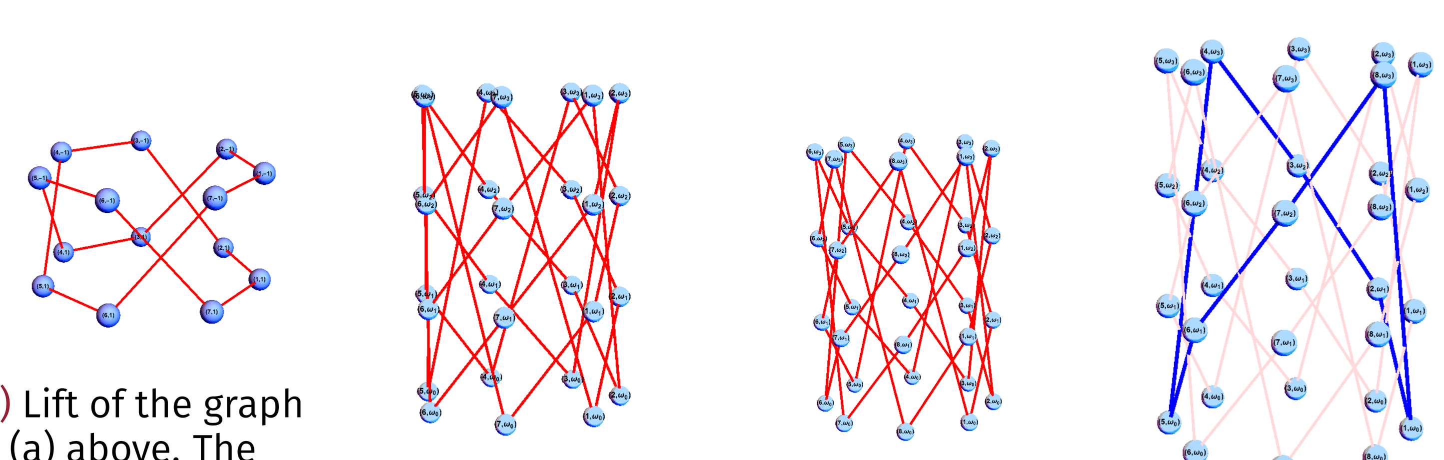
(c) 8-vertex cycle graph with complex-valued signature. All edges have signature $e^{\frac{i\pi}{2}}$, illustrated by the angular offset of the blue arrow from the red edges.

Figure: Three magnetic cycle graphs. Examples (a) and (b) are unbalanced, and (c) is balanced.

A magnetic graph (G, σ) is **balanced** if the product of the signature values along any cycle is 1; otherwise, a magnetic graph is called **unbalanced**.

Magnetic lift graphs

If (G, σ) is a magnetic graph and σ takes values in a finite subgroup $\mathbf{S}^1_p \leq \mathbf{S}^1$, we may construct a **magnetic lift graph** $\widehat{G}$ via the vertex set $V(\widehat{G}) := V(G) \times \mathbf{S}^1_p$ with two vertices (u, ω_1), (v, ω_2) adjacent if and only if $u \sim v$ and $\omega_2 = \omega_1 \sigma_{uv}$.



(a) Lift of the graph in (a) above. The lower and upper levels correspond to the signature values of +1 and -1 resp.

(b) Lift of graph (b) above; notice the 4 'levels' and connectedness

(c) Lift of graph (c) above, notice the disconnectedness of the graph.

(d) Lift of graph (c) above with one cycle highlighted.

Figure: Various lifts from the preceding magnetic graphs.

Balanced magnetic graphs always have disconnected lift graphs; unbalanced magnetic graphs usually have connected lift graphs.

What is optimal transport on graphs?

Let $G = (V(G), E(G))$ be a simple connected graph equipped with the shortest-path metric d. Suppose one has two mass (probability) distributions defined on the vertices of a graph, say $\nu, \mu : V(G) \to \mathbb{R}$, then we may consider the question of how one can transport the mass μ to the mass ν. This is formalized with the notion of a **transport plan** γ, a non-negative function which quantifies the amount of mass moved from vertex u to vertex v. $\Gamma(\mu, \nu)$ is the set of all admissible μ, ν-transport plans γ. Then the **transport cost** of μ and ν with respect to the metric d (Or the 1-**Wasserstein metric**) may be formulated:

$$W_1(\mu,\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u,v)\gamma(u,v). \quad (1)$$

Optimal transport on graphs is the study of this quantity, others like it, and the transport plans which attain them.

Let $u_0 \in V(G)$ be a fixed 'base vertex.' We define the **Lipschitz space** and its norm:

$$\mathrm{Lip}_0(G) := \left\{ f : V \to \mathbb{R} \;\middle|\; f(u_0) = 0 \right\}, \qquad \|f\|_{\mathrm{Lip}} = \max_{u \sim v} |f(u) - f(v)|$$

for each $f \in \mathrm{Lip}_0(G)$. If $f \in \mathrm{Lip}_0(G)$ with $\|f\|_{\mathrm{Lip}} \leq 1$, then f is called an **extreme point** of the unit ball in $\mathrm{Lip}_0(G)$ (denoted B_{Lip}) provided that for any $g \in \mathrm{Lip}_0(G)$, if it holds that

$$\{f + tg \;|\; t \in [-1,1]\} \subset B_{\mathrm{Lip}},$$

then $g \equiv 0$. If $\{u, v\} \in E(G)$, we say that $\{u, v\}$ is **satisfied** by f provided $|f(u) - f(v)| = 1$.

Classical convex extreme points.

Let $G = (V(G), E(G))$ be a connected simple graph, and $f \in B_{\mathrm{Lip}} \subset \mathrm{Lip}_0(G)$. Consider the subgraph H_f in G formed by $V(H_f) = V(G)$, and

$$E(H_f) := \left\{ \{u,v\} \in E(G) \;\middle|\; \{u,v\} \text{ is satisfied by } f \right\}.$$

Then f is an extreme point of B_{Lip} if and only if H_f is connected.

Separately, we define for each pair of adjacent vertices $u \sim v$ the **combinatorial atom** $m_{uv} : V(G) \to \mathbb{R}$ defined by

$$m_{uv}(x) := \mathbb{1}_{\{u\}} - \mathbb{1}_{\{v\}}$$

We define the **Arens-Eells space** to be

$$Æ(G) := \mathrm{span}_{\mathbb{R}}\{m_{uv}\}_{u \sim v}$$

equipped with the norm

$$\|m\|_{Æ} := \inf \left\{ \sum_i |a_i| \;\middle|\; m = \sum_i a_i m_{u_i v_i} \right\}.$$

Classical Kantorovich Duality on Graphs.

The spaces $Æ(G)^*$ and $\mathrm{Lip}_0(G)$ are isometrically isomorphic. It holds

$$W_1(\mu,\nu) = \sup \left\{ \left| \sum_{u \in V(G)} f(u)(\mu(u) - \nu(u)) \right| \;\middle|\; f \in \mathrm{Lip}_0(G), \|f\|_{\mathrm{Lip}} \leq 1 \right\}$$
$$= \|\mu - \nu\|_{Æ}$$

Open Questions

(1) How can we further describe $\|\cdot\|_{Æ^\sigma}$ in terms of the norm $\|\cdot\|_{Æ}$ using the compression mapping?
(2) How can magnetic transport be interpreted as a physical process?

Notation

- V^* algebraic dual space
- $\bar{z}$ complex conjugate
- G simple connected graph
- $\mathbf{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$
- $\mathbf{S}^1_p$ p-th roots of unity
- $\mathrm{span}_{\mathbb{F}}\{\dots\}$ $\mathbb{F}$-linear span of $\{\dots\}$

Results

In the case of a simple magnetic graph (G, σ), we may consider two new normed spaces. The σ-**Lipschitz** space $\mathrm{Lip}^\sigma(G)$ and its norm are defined by

$$\mathrm{Lip}^\sigma(G) := \{f : V(G) \to \mathbb{C}\}, \qquad \|f\|_{\mathrm{Lip}^\sigma} = \max_{u \sim v} |f(u) - \sigma_{uv} f(v)|.$$

If $f \in \mathrm{Lip}^\sigma(G)$ with $\|f\|_{\mathrm{Lip}^\sigma} \leq 1$, then f is called an **extreme point** of the unit ball in $\mathrm{Lip}^\sigma(G)$ (denoted B_{Lip^σ}) provided that for any $g \in \mathrm{Lip}^\sigma(G)$, if it holds that

$$\{f + tg \;|\; t \in [-1,1]\} \subset B_{\mathrm{Lip}^\sigma},$$

then $g \equiv 0$. If $\{u, v\} \in E(G)$, we say that $\{u, v\}$ is $\sigma-$**satisfied**$byfprovided$|f(u)-$\sigma_{uv} f(v)| = 1$.

Convex extreme points.

Let (G, σ) be an unbalanced graph, and $f \in B_{\mathrm{Lip}^\sigma}$. Then f is an extreme point of B_{Lip^σ} if and only if the magnetic graph H_f defined by the vertex set $V(G)$, the edge set

$$E(H_f) := \left\{ \{u,v\} \in E(G) \;\middle|\; \{u,v\} \text{ is } \sigma\text{-satisfied by } f \right\},$$

and which we equip with the same signature structure σ as on G, is unbalanced on each of its connected components.

Similarly, we may define a **magnetic atom** for every pair of adjacent vertices u, v, and the σ-**Arens-Eells space** to be

$$m^\sigma_{uv}(x) := \mathbb{1}_{\{u\}} - \sigma_{uv}\mathbb{1}_{\{v\}}, \qquad Æ^\sigma(G) := \mathrm{span}_{\mathbb{C}}\{m^\sigma_{uv}\}_{u \sim v}$$

equipped with the norm

$$\|m\|_{Æ^\sigma} := \inf \left\{ \sum_i |a_i| \;\middle|\; m = \sum_i a_i m^\sigma_{u_i v_i} \right\}.$$

Kantorovich duality.

For an unbalanced, simple magnetic graph (G, σ) the spaces $Æ^\sigma(X)$ and $\mathrm{Lip}^\sigma(X)^*$ are isometrically isomorphic.

Compression Transformation

We define the linear compression mapping $C : Æ(\widehat{G}) \to Æ^\sigma(G)$ by setting, for each $m \in Æ(\widehat{G}), u \in V(G)$,

$$(Cm)(u) = \sum_{\xi \in \mathbf{S}^1_p} \xi m(u, \xi).$$

C is in fact a surjective contraction onto the space $Æ^\sigma(G)$. We have the equation

$$\|m^\sigma\|_{Æ^\sigma} = \min \left\{ \|m\|_{Æ} \middle| m \in Æ(\widehat{X}); Cm = m^\sigma \right\}$$

for each $m \in Æ^\sigma(G)$.

References.

[1] Solomon, Justin (2018). "Optimal Transport on Discrete Domains." Notes for AMS Short Course on Discrete Differential Geometry, San Diego.
[2] Weaver, Nik (1999). "Lipschitz algebras." World Scientific, River Edge, N.J.