Kantorovich Duality and Signature Balance for Generalized Signed Graphs

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Talk Highlights and Objectives

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1. **Background** Explain the notions of signed graphs and signature balance in both concrete and abstract settings
2. **Result** Give some characterizations of signature balance in the abstract setting
3. **Background** Give a short primer on the well-established Kantorovich duality for graphs
4. **Result** Find sufficient conditions on a signed graph for a specialized Kantorovich-type duality to hold in the abstract setting
5. **Questions & Discussion**

Signed Graphs: Abstract Case

1. Throughout, $G = (V, E)$ is a **connected** combinatorial (undirected) graph. We assume no loops or multiple edges, and denote adjacency with a tilde.
2. The **oriented edge set** of a graph G is given by

$$E^{\mathrm{or}}(G) := \{(u, v), (v, u) : u, v \in V(G), u \sim v\}.$$

3. Γ will be used to denote a general group; its identity element denoted e. Γ will act on a **Banach space** X via a **left action** α.
4. A **signature** on G is a map

$$\sigma : E^{\mathrm{or}}(G) \to \Gamma : (u, v) \mapsto \sigma_{uv},$$

   satisfying the property $\sigma_{vu} = (\sigma_{uv})^{-1}$.
5. A pair (G, σ) is called a **signed graph**.
6. If $\tau : V \to \Gamma$ is some function and ρ is any signature, then we may produce the τ-switched signature denoted ρ^τ via

$$\rho^\tau_{uv} := \tau(u)\rho_{uv}\tau(v)^{-1}. \tag{1}$$

   Two signatures related in this way are called **switching equivalent**.

Some concrete examples

1. A **magnetic graph** is a pair (H, ρ) where H is a graph and ρ is a signature taking values in the group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$.
2. The signature is this case is acting on $\mathbb{C}$; in turn, functions $V \to \mathbb{C}$.
3. In the past, signature structures on these types of graphs have served to discretely model magnetic fields or quantum mechanical systems [Lieb and Loss(1993)].
4. A **connection graph** is a pair (F, ω) where F is a graph and ω is a signature taking values in the real orthogonal group $O_n(\mathbb{R})$.
5. The signature is in this case is acting on $\mathbb{R}^n$; in turn, functions $V \to \mathbb{R}^n$.
6. These types of signatures have been used to model 3D structures by synthesizing 2D images of various faces of the structrues and the rotations which relate the perspectives from which the images were captured.

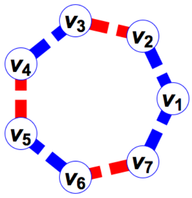

Figure: An example of a magnetic graph with ± 1 signature.

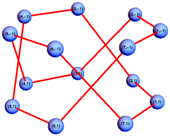

Figure: The **lift** of the preceding magnetic graph. Omitted here, lifts can turn combinatorial properties of a signature into structural and geometric properties of the associated lift graph.

Balanced signatures: Concrete case

Proposition

Let (H, ρ) be a magnetic graph. The following are equivalent:

- *For every oriented cycle expressed as a list of incident oriented edges in the form $((u_0, u_1), (u_1, u_2), \ldots, (u_{n-1}, u_n = u_0))$, it holds $\prod_{i=0}^{n-1} \rho_{u_i u_{i+1}} = 1$.*
- *ρ is switching equivalent to the trivial signature.*

Proposition

Let (F, ω) be a connection graph. Then the following are equivalent:

- *For every oriented cycle expressed as a list of incident oriented edges in the form $((u_0, u_1), (u_1, u_2), \ldots, (u_{n-1}, u_n = u_0))$, it holds $\prod_{i=0}^{n-1} \omega_{u_i u_{i+1}} = Id_n$, the $n \times n$ identity matrix.*
- *ω is switching equivalent to the trivial signature, in the sense that there exists $T : V \to O_n(\mathbb{R})$ such that*

  $$T(u)\omega_{uv}T(v)^{-1} = Id_n$$

  for each $u \sim v$.

1. Signatures which satisfy either of the preceding conditions are called **balanced**.
2. **Question** How do we generalize this notion to the general case where the signature takes values in an arbitrary group?

Balanced signatures: The abstract case

Theorem (Signature equivalence (a))

Let $G = (V, E)$ be a connected simple graph, σ a signature on G taking values in a group Γ, X a Banach space, and α a left action of of Γ on X. Then the following are equivalent:

- *There exists a function $f : V \to X$, not identically 0, such that for every oriented edge $(u, v) \in E^{or}(G)$, it holds*

$$f(u) = \alpha(\sigma_{uv})f(v).$$

- *There exists a nonzero element $x \in X$ and some $u_0 \in V$ such that for each directed cycle of the form*

$$((u_0, u_1), (u_1, u_2), \ldots, (u_{n-2}, u_{n-1}), (u_{n-1}, u_n = u_0)) \subset E^{or}(G)$$

it holds

$$\alpha\left(\prod_{i=0}^{n-1} \sigma_{u_i u_{i+1}}\right) x = x.$$

1. The first condition generalizes the preceding notion of switching equivalence to the trivial signature
2. The second condition generalizes the notion of trivial signature products along cycles; here, the triviality is reflected by the stability of the element x under the action of the signature product along the cycle.

Terminology

1. Recalling the left action α of Γ on X, let X^* be the continuous dual to X. We can define the **induced action** α^* on X^* by the equation

$$(\alpha^*(\psi))(x) = \psi(\alpha(x))$$

   for each $\psi \in X^*$ and $x \in X$. While not a true group action in some sense, we can work with it in the same manner.
2. Having fixed a group, the 4-tuple (G, σ, α, X) is said to be **balanced** under the action of α on X provided any (and hence all) of the preceding conditions hold.
3. Having fixed a group, the 4-tuple (G, σ, α, X) is said to be $*$-**balanced** under the action of α^* on X^* provided $(G, \sigma, \alpha^*, X^*)$ is balanced.

Background: Classical Kantorovich Duality on Graphs

1. Let $u_0 \in V(G)$ be a fixed 'base vertex.' We define the **Lipschitz space** and its norm:

$$\mathrm{Lip}_0(G) := \left\{ f : V \to \mathbb{R} \;\middle|\; f(u_0) = 0 \right\}, \qquad \|f\|_{\mathrm{Lip}} = \max_{u \sim v} |f(u) - f(v)|$$

   for each $f \in \mathrm{Lip}_0(G)$.
2. Separately, we define for each pair of adjacent vertices $u \sim v$ the **combinatorial atom** $m_{uv} : V(G) \to \mathbb{R}$ defined by

$$m_{uv}(x) := \mathbb{I}_{\{u\}} - \mathbb{I}_{\{v\}}.$$

3. We define the **Arens-Eells space** to be

$$Æ(G) := \mathrm{span}_{\mathbb{R}} \{m_{uv}\}_{u \sim v}$$

   equipped with the norm

$$\|m\|_{Æ} := \inf \left\{ \sum_i |a_i| \;\middle|\; m = \sum_i a_i m_{u_i v_i} \right\}.$$

Theorem (Kantorovich duality, 1940s)

The spaces $Æ(G)^*$ and $Lip_0(G)$ are isometrically isomorphic. It holds for any two probability densities μ, ν defined on the vertices of G,

$$\|\mu - \nu\|_{Æ} = \sup \left\{ \left| \sum_{u \in V(G)} f(u)(\mu(u) - \nu(u)) \right| \;\middle|\; f \in Lip_0(G), \|f\|_{Lip} \le 1 \right\}$$

Signed Lipschitz Space

Definition

The signed Lipschitz space is given by

$$\mathrm{Lip}^{\sigma}(G;X) := \{f : V \to X \;\big|\; \exists C \geq 0 \text{ s.t.} \|f(u) - \alpha(\sigma_{uv})f(v)\|_X \leq C \;\; \forall u \sim v\}.$$

equipped with the semi-norm

$$\|f\|_{\mathrm{Lip}^{\sigma}(G;X)} := \max_{u \sim v} \|f(u) - \alpha(\sigma_{uv})f(v)\|_X.$$

Lemma

If (G, σ, α, X) is not balanced, $\mathrm{Lip}^{\sigma}(G;X)$ will be a Banach space.

Signed Arens-Eells Space

Definition

Letting $a \in X$ and $u, v \in V$ be any adjacent vertices, we can define $am^{\sigma}_{uv} : V \to X$ via

$$am^{\sigma}_{uv}(w) = \begin{cases} a & \text{if } w = u \\ -\alpha(\sigma_{uv})a & \text{if } w = v \\ 0 & \text{otherwise} \end{cases}, \quad \forall w \in V.$$

along with the signed Arens-Eells space

$$\text{Æ}^{\sigma}(G; X) := \{am^{\sigma}_{uv} \;\big|\; u, v \in V, u \sim v, a \in X\}$$

equipped with the norm defined by

$$\|m\|_{\text{Æ}^{\sigma}(G;X)} = \inf\left\{\sum_{i=1}^{n} \|a_i\|_X \;\Big|\; m = \sum_{i=1}^{n} a_i m^{\sigma}_{u_i v_i}, \quad u_i \sim v_i, a_i \in X, \; 1 \le i \le n\right\}.$$

Lemma

If (G, σ, α, X) is not $*$-balanced, $\text{Æ}^{\sigma}(G; X) = Lip^{\sigma}(G; X)$ as vector spaces.

Abstract Kantorovich Duality

Theorem

If (G, σ, α, X) *is not* $*$*-balanced, then we have the identification*

$$Æ^{\sigma}(G; X)^{*} \equiv Lip^{\sigma}(G; X^{*}).$$

Theorem

If (G, σ, α, X) *is not balanced, then we have the identification*

$$Æ^{\sigma}(G; X^{*}) \equiv Lip^{\sigma}(G; X)^{*}.$$

1. **The punchline?**
2. The Kantorvich duality we know and love on classical graphs (and more generally, metric spaces) holds on signed graphs with a broad degree of generality in the signature type and function spaces; however, there is some subtlety in the requirements of the signature to ensure the duality does indeed occur.

Questions & Discussion

Key References

Carsten Lange, Shiping Liu, Norbert Peyerimhoff, and Olaf Post.
Frustration index and Cheeger inequalities for discrete and continuous magnetic Laplacians.
Calc. Var. Partial Differential Equations, 54(4), 2015.

Elliott H. Lieb and Michael Loss.
Fluxes, Laplacians, and Kasteleyn's theorem.
Duke Math. J., 71(2), 1993.

Robert E. Megginson.
An introduction to Banach space theory, volume 183 of *Graduate Texts in Mathematics*.
Springer-Verlag, New York, 1998.

Nik Weaver.
Lipschitz algebras.
World Scientific Publishing Co., Inc., River Edge, NJ, 1999.

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