Kantorovich Duality & Optimal Transport Problems on Magnetic Graphs

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Objectives

1. Explain the concepts of magnetic graphs and their ‘lifts’
2. State a classical Kantorovich duality result and introduce a new formulation for magnetic graphs
3. Characterize the extreme points in Lipschitz-type function spaces for both classical and magnetic graphs
4. Present a ‘compression equation’

Magnetic Graphs

1. Throughout, $G = (V(G), E(G))$ is a **simple** and **connected** combinatorial (undirected) graph.
2. The **oriented edge set** of a graph G is given by

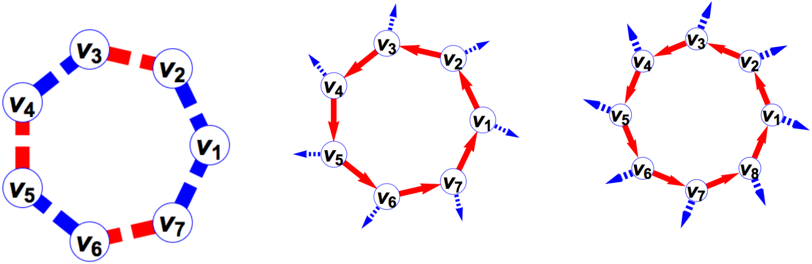
$$E^{\mathrm{or}}(G) := \{(u, v), (v, u) : u, v \in V(G), u \sim v\}.$$

3. A **signature** on a graph is a map

$$\sigma : E^{\mathrm{or}}(G) \to \{z \in \mathbb{C} : |z| = 1\} : (u, v) \mapsto \sigma_{uv},$$

   satisfying the property $\sigma_{vu} = \overline{\sigma_{uv}}$.
4. A pair (G, σ) is called a **magnetic graph**.
5. A magnetic graph (G, σ) is **balanced** if the product of the signature values along any directed cycle is 1; otherwise, a magnetic graph is called **unbalanced**.

Some examples...



(a) 7-vertex cycle graph, with real-valued signature. The edges with positive signature are in blue, those with negative signature are in red.

(b) 7-vertex cycle graph with complex-valued signature. All edges have signature $e^{\frac{i\pi}{2}}$, illustrated by the angular offset of the blue arrow from the red edges.

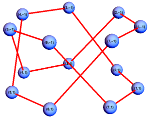
(c) 8-vertex cycle graph with complex-valued signature. All edges have signature $e^{\frac{i\pi}{2}}$, illustrated by the angular offset of the blue arrow from the red edges.

Figure: Three magnetic cycle graphs. Examples (a) and (b) are unbalanced, and (c) is balanced.

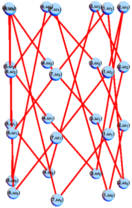
Magnetic lift graphs

1. If (G, σ) is a magnetic graph and σ takes values in a group of the p-th roots of unity $\mathbf{S}^1_p$, we may construct a **magnetic lift graph** $\widehat{G}$ via the vertex set $V(\widehat{G}) := V(G) \times \mathbf{S}^1_p$, with two vertices (u, ω_1), (v, ω_2) adjacent if and only if $u \sim v$ and $\omega_2 = \omega_1 \sigma_{uv}$.
2. Balanced magnetic graphs always have disconnected lift graphs; unbalanced magnetic graphs usually have connected lift graphs.

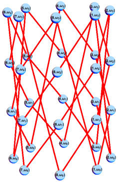
More examples...



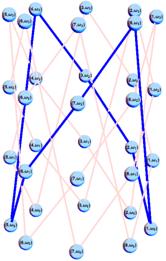
(a) Lift of the graph in (a) above. The lower and upper levels correspond to the signature values of +1 and -1 resp.



(b) Lift of graph (b) above; notice the 4 'levels' and connectedness



(c) Lift of graph (c) above, notice the disconnectedness of the graph.



(d) Lift of graph (c) above with one cycle highlighted.

Figure: Various lifts from the preceding magnetic graphs.

Classical OT on Graphs

1. Let $G = (V(G), E(G))$ be a simple connected graph equipped with the shortest-path metric d.
2. Suppose one has two mass (probability) distributions defined on the vertices of a graph, say $\nu, \mu : V(G) \to \mathbb{R}$, then we may consider the question of how one can transport the initial mass distribution μ to the terminal mass distribution ν.
3. This is formalized with the notion of a **transport plan** $\gamma : V \times V \to \mathbb{R}$, a non-negative function which quantifies the amount of mass moved from vertex u to vertex v. $\Gamma(\mu, \nu)$ is the set of all admissible μ, ν-transport plans.
4. Then the **transport cost** of μ and ν with respect to the metric d (Or the 1-**Wasserstein metric**) may be formulated:

$$W_1(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \sum_{u \in V(G)} \sum_{v \in V(G)} \gamma(u, v) d(u, v). \tag{1}$$

5. **Optimal transport on graphs** is the study of this quantity, others like it, and the transport plans which attain them.

Classical Kantorovich Duality (1)

1. Let $u_0 \in V(G)$ be a fixed 'base vertex.' We define the **Lipschitz space** and its norm:

$$\mathrm{Lip}_0(G) := \left\{ f : V \to \mathbb{R} \;\middle|\; f(u_0) = 0 \right\}, \qquad \|f\|_{\mathrm{Lip}} = \max_{u \sim v} |f(u) - f(v)|$$

   for each $f \in \mathrm{Lip}_0(G)$.

2. Separately, we define for each pair of adjacent vertices $u \sim v$ the **combinatorial atom** $m_{uv} : V(G) \to \mathbb{R}$ defined by

$$m_{uv}(x) := \mathbb{I}_{\{u\}} - \mathbb{I}_{\{v\}}.$$

3. We define the **Arens-Eells space** to be

$$\text{Æ}(G) := \mathrm{span}_{\mathbb{R}} \{m_{uv}\}_{u \sim v}$$

   equipped with the norm

$$\|m\|_{\text{Æ}} := \inf \left\{ \sum_i |a_i| \;\middle|\; m = \sum_i a_i m_{u_i v_i} \right\}.$$

Classical Kantorovich Duality (2)

Theorem (Kantorovich duality, 1940s [4])

The spaces $Æ(G)^*$ and $Lip_0(G)$ are isometrically isomorphic. It holds

$$W_1(\mu,\nu) = \sup\left\{ \left| \sum_{u\in V(G)} f(u)(\mu(u)-\nu(u)) \right| \;\middle|\; f \in Lip_0(G), \|f\|_{Lip} \leq 1 \right\}$$
$$= \|\mu-\nu\|_{Æ}$$

1. Note that the transport cost $W_1(\mu,\nu)$ we are interested in is now formulated as the norm $\|\mu-\nu\|_{Æ}$.
2. Further note that the sup expression above may be restricted to those $f \in \mathrm{Lip}_0(G)$ which are convex extreme points of the unit ball in the space $\mathrm{Lip}_0(G)$.

Magnetic Kantorovich Duality

1. The σ-**Lipschitz** space $\mathrm{Lip}^\sigma(G)$ and its norm are defined by

$$\mathrm{Lip}^\sigma(G) := \{f : V(G) \to \mathbb{C}\}, \qquad \|f\|_{\mathrm{Lip}^\sigma} = \max_{u\sim v} |f(u) - \sigma_{uv} f(v)|.$$

2. Similarly, we may define a **magnetic atom** for every pair of adjacent vertices u, v, and the σ-**Arens-Eells space** to be

$$m^\sigma_{uv}(x) := \mathbb{I}_{\{u\}} - \sigma_{uv}\mathbb{I}_{\{v\}}, \qquad Æ^\sigma(G) := \mathrm{span}_{\mathbb{C}}\{m^\sigma_{uv}\}_{u\sim v}$$

3. equipped with the norm

$$\|m\|_{Æ^\sigma} := \inf\left\{ \sum_i |a_i| \;\middle|\; m = \sum_i a_i m^\sigma_{u_i v_i} \right\}.$$

Theorem (Kantorovich duality, magnetic version, SR 2018)

For an unbalanced, simple magnetic graph (G, σ) the spaces $Æ^\sigma(X)$ and $\mathrm{Lip}^\sigma(X)^*$ are isometrically isomorphic.

Classical Extreme Points

1. If $f \in \mathrm{Lip}_0(G)$ with $\|f\|_{\mathrm{Lip}} \leq 1$, then f is called an **extreme point** of the unit ball in $\mathrm{Lip}_0(G)$ (denoted B_{Lip}) provided that for any $g \in \mathrm{Lip}_0(G)$, if it holds that
$$\left\{f + tg \;\middle|\; t \in [-1,1]\right\} \subset B_{\mathrm{Lip}},$$
then $g \equiv 0$.
2. If $\{u, v\} \in E(G)$, we say that $\{u, v\}$ is **satisfied** by f provided $|f(u) - f(v)| = 1$.

Theorem (Classical extreme points, 1990s [1])

Let $G = (V(G), E(G))$ be a connected simple graph, and $f \in B_{Lip} \subset Lip_0(G)$. Consider the subgraph H_f in G formed by $V(H_f) = V(G)$, and

$$E(H_f) := \left\{\{u, v\} \in E(G) \;\middle|\; \{u, v\} \text{ is satisfied by } f\right\}.$$

Then f is an extreme point of B_{Lip} if and only if H_f is connected.

Magnetic Extreme Points

1. If $f \in \mathrm{Lip}^\sigma(G)$ with $\|f\|_{\mathrm{Lip}^\sigma} \leq 1$, then f is called an **extreme point** of the unit ball in $\mathrm{Lip}^\sigma(G)$ (denoted B_{Lip^σ}) provided that for any $g \in \mathrm{Lip}^\sigma(G)$, if it holds that
$$\left\{f + tg \;\middle|\; t \in [-1, 1]\right\} \subset B_{\mathrm{Lip}^\sigma},$$
then $g \equiv 0$.
2. If $\{u, v\} \in E(G)$, we say that $\{u, v\}$ is σ-**satisfied** by f provided $|f(u) - \sigma_{uv} f(v)| = 1$.

Theorem (Magnetic extreme points, SR 2018)

Let (G, σ) be an unbalanced graph, and $f \in B_{Lip^\sigma}$. Then f is an extreme point of B_{Lip^σ} if and only if the magnetic graph H_f defined by the vertex set $V(G)$, the edge set

$$E(H_f) := \left\{\{u, v\} \in E(G) \;\middle|\; \{u, v\} \text{ is } \sigma\text{-satisfied by } f\right\},$$

and which we equip with the same signature structure σ as on G, is unbalanced on each of its connected components.

Compression Equation

1. We wish to somehow relate the σ-Arens-Eells norm for functions on a magnetic graph (G, σ) to the classical Arens-Eells norm for functions on the lift graph $\widehat{G}$.
2. We define the **linear compression mapping** $C : Æ(\widehat{G}) \to Æ^{\sigma}(G)$ by setting, for each $m \in Æ(\widehat{G}), u \in V(G)$,

$$(Cm)(u) = \sum_{\xi \in \mathbf{S}_p^1} \xi m(u, \xi).$$

3. C is in fact a surjective contraction onto the space $Æ^{\sigma}(G)$.

Theorem (Compression equation, SR 2018)

We have the equation

$$\|m^{\sigma}\|_{Æ^{\sigma}} = \min\left\{\|m\|_{Æ} \;\middle|\; m \in Æ(\widehat{X}); Cm = m^{\sigma}\right\}$$

for each $m \in Æ^{\sigma}(G)$.

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