Boundary Value Problems and Green’s Functions on Magnetic Graphs
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What is a magnetic graph?
Let \( G = (V, E) \) be a finite simple graph, with vertex set \( V \) and an edge set \( E \). The set of oriented edges is given by \( E^\sigma(G) = \{(u, v) \in (V, E) : (u, v) \in E \} \).

A signature on \( G \) is a map \( \sigma : E^\sigma(G) \to \{-1, 1\} \) satisfying \( \sigma_{uv} = -\sigma_{vu} \). A signed or magnetic graph is a graph \( G \) equipped with a signature.

Discrete Laplacians
Graphs serve as extremely useful discrete analogues of continuous domains, often serving as good settings for numerical approximations to solutions of partial differential equations. We investigate the discrete cousin of the classical Laplacian operator, in both unsigned and signed structures and problems arise in many physical models where discrete domains (namely, graphs) can more efficiently describe continuous regions; in particular, those of quantum mechanics, where a signature structure helps to describe atomic structures with the presence of magnetic potential.

Illustrating the Discrete Dirichlet Problem using Mathematica
We illustrate solutions to the combinatorial and magnetic Dirichlet problems subjected to the same boundary condition.

Result 1: Solution to the Magnetic Poisson Problem
Suppose \( G \) is a finite, simple signed graph, and \( H \) is a proper, connected subgraph of \( G \). Given \( f \in L^2(V(H)) \), we wish to find a function \( u \in L^2(V(H)) \) so that

\[
\begin{align*}
(L^\sigma f)(v) &= g(v) & v \in V(H), \\
u(v) &= f(v) & v \in V(H).
\end{align*}
\]

We obtain \( u \) by finding \( u_1, u_2 \in L^2(V(H)) \) which solve

\[
\begin{align*}
(L^\sigma u_1)(v) &= 0 & v \in V(H), \\
u_1(v) &= f(v) & v \in V(H), \\
(L^\sigma u_2)(v) &= g(v) & v \in V(H), \\
u_2(v) &= f(v) & v \in V(H).
\end{align*}
\]

so that \( u = u_1 - u_2 \). Under our assumptions on the domains, unique solutions will exist. We have two theorems giving constructions of the solutions.

Theorem
Let \( \{\phi_i\}_{i \in \mathbb{Z}} \) be an orthonormal basis of \( L^2(V(H)) \) of eigenvectors of \( L^\sigma \), with associated eigenvalues \( \{\lambda_i\}_{i \in \mathbb{Z}} \). We extend each \( \phi_i \) to \( \tilde{\phi}_i \) agreeing with \( \phi_i \) on \( V(H) \) and \( \tilde{\phi}_i \equiv 0 \) on \( V(H) \) for \( i \leq m \).

The solution to (3) is given by

\[
u_i(z) = \sum_{m \in \mathbb{Z}} \lambda_i^{m/2} \frac{\tilde{\phi}_i(z)}{\tilde{\phi}_i(0)} z \in V(H).
\]

The matrix \( L^\sigma \) is invertible, and the solution to (4) is given by

\[
u_i(z) = \left( L^\sigma^{-1} \right)_i^0 z \in V(H),
\]

The matrix \( L^\sigma \) is a magnetic Green’s function, in the sense that it is fundamental representation tool in the solution boundary value problem, both theoretically and in practice.

Result 2: Magnetic Green Identities
We develop two discrete Green’s identities, which were fundamental in the proofs of the preceding theorems. Let \( G \) be as in the previous result.

Theorem
Let \( f, g \in L^2(V(H)) \). We have

\[
\sum_{v \in V(H)} \left((L^\sigma f)(v) - f(v)\right) (g(v) - g(0)) = \sum_{v \in V(H)} f(v) (L^\sigma g)(v) - \sum_{v \in V(H)} (L^\sigma f)(v) g(v).
\]

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\]

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References

Abstract
Let \( G = (V, E) \) be a finite simple graph. We impose on \( G \) the additional structure of a signature, a function which maps edges into the set of complex numbers of modulus 1. This induces a second-order difference operator for complex-valued functions defined on the vertex set of \( G \) which is a discrete analog of the classical Laplacian, and consequently discrete boundary value problems on proper and sufficiently connected subgraphs of \( G \). We construct a solution to Poisson type problems and explore some applications, including the role of discrete Green’s functions in constructions of solutions. These problems and structures arise in many physical models where discrete domains (namely, graphs) can more efficiently describe continuous regions; in particular, those of quantum mechanics, where a signature structure helps to describe atomic structures with the presence of magnetic potential.

Figure: The graph \( G \), with \( V(H) \) in yellow, and negatively signed edges dashed.

Figure: The boundary function \( f \), with discrete plot points joined to form a curve.

Figure: The solution \( u \), defined on the whole lattice, with plot points joined as a surface.

Figure: The solution \( v \), defined on the whole lattice, with plot points joined as a surface.