Sketch of proof that **AIC is asymptotically unbiased for AI**:

That is, we want to show:

$$E\{l(\hat{\theta})\} - p = E[E_{y^*}\{l(\hat{\theta}; y^*)\}] + o(1).$$

We will show:

$$E\{l(\hat{\theta})\} - \mathrm{tr}\left\{\Omega \cdot \mathrm{Var}(\hat{\theta})\right\} = E[E_{y^*}\{l(\hat{\theta}; y^*)\}] + o(1),$$

where $\Omega = -E\{l''(\theta_0)\}$.

Consider the Taylor series expansion:

$$l(\hat{\theta}) = l(\theta_0) + (\hat{\theta} - \theta_0)^\top l'(\theta_0) + \frac{1}{2}(\hat{\theta} - \theta_0)^\top l''(\bar{\theta})(\hat{\theta} - \theta_0),$$

where $\bar{\theta}$ is between θ_0 and $\hat{\theta}$.

Another Taylor series expansion:

$$l'(\theta_0) = -l''(\tilde{\theta})(\hat{\theta} - \theta_0),$$

where $\tilde{\theta}$ is between θ_0 and $\hat{\theta}$. (Note $\hat{\beta}$ is MLE so $l'(\hat{\theta}) = 0$.)

Therefore

$$\begin{aligned} l(\hat{\theta}) &= l(\theta_0) - \frac{1}{2}(\hat{\theta} - \theta_0)^\top l''(\theta_0)(\hat{\theta} - \theta_0) + o_p(1), \\ &= l(\theta_0) + \frac{1}{2}(\hat{\theta} - \theta_0)^\top \Omega(\hat{\theta} - \theta_0) + o_p(1) \end{aligned} \quad (1)$$

Now from (**??**),

$$\begin{aligned} E\{l(\hat{\theta})\} &= E\{l(\theta_0)\} + \frac{1}{2}E\{(\hat{\theta} - \theta_0)^\top \Omega(\hat{\theta} - \theta_0)\} + E\{o_p(1)\} \\ &= E\{l(\theta_0)\} + \frac{1}{2}\mathrm{tr}\left\{\Omega \cdot \mathrm{Var}(\hat{\theta})\right\} + o(1), \end{aligned}$$

assuming that the $o_p(1)$ term is uniformly integrable to get $E\{o_p(1)\} = o(1)$.

Similar to the above, we can show that (Ex.)

$$E[E_{y^*}\{l(\hat{\theta}; y^*)\}] = E\{l(\theta_0)\} - \frac{1}{2}\text{tr}\left\{\Omega \cdot \text{Var}(\hat{\theta})\right\} + o(1).$$

And this completes the proof.

Note: what was used above was the consistency of $\hat{\theta}$ and the second order Taylor expansion of the log-likelihood.