Math 173A Discussions 5-6

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1. Define $S = \{x \in \mathbb{R}^n : l \leq x \leq u\}$ where the inequalities are defined component-wise.

   (a) Writing $l = (l_1, \ldots, l_n)^T$ and $u = (u_1, \ldots, u_n)^T$, explain why solving

   $$\begin{aligned} \min_{x \in \mathbb{R}} \quad & |x - z| \\ \text{s.t.} \quad & l_i \leq x \leq u_i \end{aligned}$$

   for $i = 1, \ldots, n$ is sufficient to solve for

   $$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & ||x - z|| \\ \text{s.t.} \quad & l \leq x \leq u \end{aligned}$$

   (b) Give a closed form expression for the projection of $z \in \mathbb{R}^n$ onto S. Carefully justify each step.

2. The statement of (a special case of) the Hilbert Projection Theorem is that for a nonempty closed convex subset C of $\mathbb{R}^n$, the projection of any vector $x \in \mathbb{R}^n$ onto C exists and is unique.

   (a) Show that existence does not need to hold if the subset C is not closed.

   (b) Show that uniqueness does not need to hold if the subset C is not convex.

   (c) Prove the uniqueness assertion of the Hilbert Projection Theorem on $\mathbb{R}^n$.
   Hint: You just showed that you need to use convexity.

3. Suppose that, as part of your minimization algorithm, your computer needs to solve the linear system

   $$\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   (a) Show that when $\epsilon \neq 1$, the unique solution is given by

   $$x_1 = -x_2 = \frac{1}{\epsilon - 1}$$

   (b) Suppose your computer is precise to only 6 digits. Set $\epsilon = 10^{-6}$. Use Gaussian elimination and conclude that

   $$x_2 = \frac{-10^6}{1 - 10^6}$$

   What number does your computer solve x_2 as?

   (c) Your computer will use that value of x_2 and the first row of your linear system to now solve for x_1. What value does it solve x_1 as?

   (d) How does it compare to your answer from part a)?

4. Let $B \succ 0$, f be a differentiable function on $\mathbb{R}^n$ and define

   $$q(x + p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T B p$$

   Prove that any minimizer of this function must be a descent direction for f at x.