Homework 5 for Math 173A - Fall 2019

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x - a)^4$ where a is some constant. Suppose we apply Newton's method to solve this problem.
   (a) Write down the update equation for Newton's method.
   (b) Let $y^{(t)} = |x^{(t)} - a|$, where $x^{(t)}$ is hte t-th iterate in Newton's method. Show that $z^{(t+1)} = \frac{2}{3} z^{(t)}$.
   (c) Show that $x^{(t)}$ conveges to a from any initialization $x^{(0)}$.

2. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be given by
$$f(x_1, x_2, x_3, x_4) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4.$$
   (a) Find the gradient and Hessian of f.
   (b) Write the Newton update for f.

3. Let $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.
   (a) Prove that (1,1) is the unique global minimum of this function.
   (b) Starting at (0,0), apply two iterations of Newton's method (by hand). Hint: Recall the closed form expression for inverting a 2×2 matrix.
   (c) Repeat part (b) using gradient descent with step size $\mu = 0.5$.

4. **Computer Problem:** Let $f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$ as in the previous question.
   (a) Write computer code implementing Newton's method for this function.
   (b) Write computer code implementing Gradient Descent with a fixed step size $\mu = 10^{-3}$ for this function.
   (c) Write computer code implementing Gradient Descent with backtracking line-search for this function (You may select β and γ as you wish).
   (d) Starting with the same random initialization $x^{(0)}$, run each of the three algorithms above, and plot $\|x^{(t)} - x^*\|$ for each of them, on the same figure. On a separate figure, plot $f(x^{(t)}) - f(x^*)$ for each of them. Comment on the performance of these methods.

5. Suppose we wish to minimize the function $f(x_1, x_2) = x_1^2 + x_2^2$ subject to the constraint $x_1^2 + 2x_2^2 - 1 = 0$ (note that this is the equation of an ellipse).
   (a) Use the method of Lagrange multipliers to find the optimum.
   (b) Use the method of Lagrange multipliers to find the optimum if we add the constraint $x_1 + x_2 \geq 1$.