Homework 1 for Math 173A - Fall 2019

Review of calculus, and convexity

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^4 - 2x^3 + x^2$.
   (a) Find (analytically) the minima and maxima of f.
   (b) Determine which of the minima and maxima are local or global, and justify your answer.

2. Use the definition of convexity (of functions) to show that
   (a) the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x-1)^2$ is convex.
   (b) the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1^2 + x_2^2$ is convex.
   (c) the function $f : \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is *not* convex.

3. Show that
   (a) if S and T are convex sets, then $S \cap T$ is convex.
   (b) the intersection of any collection of convex sets is convex.

4. Show (using the definition of convex sets) that the following sets are convex.
   (a) $S := \{x \in \mathbb{R}^2 : 0 \leq x_1 + x_2 \leq 1\}$. Sketch the set S.
   (b) $T := \{x \in \mathbb{R}^n : \alpha \leq \langle a, x \rangle \leq \beta\}$, where α and β are real numbers and $a \in \mathbb{R}^n$. Sketch the set T when $n = 2, a = (1, 0)$ and $\alpha = 0, \beta = 1/2$.

5. Find the **gradient and hessian** of each of the following functions and use the hessians to check whether the functions are convex.
   (a) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = (x_1 - x_2)^2$.
   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 - 2xy - y^6$.
   (c) $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x_1, x_2, x_3) = \cos(x_1) + x_2 x_3^2$ (you may use a computer to find the eigenvalues of the hessian).

6. What is the direction of maximal increase, at the origin, of each of the three functions in Exercise 5 above?