Math 170B, Spring 2018
Review for Midterm 2

Problem 1. Consider the following data tables:

(I)
x	0	1	2
y	10	1	19

(II)
x	-1	0	1
y	1	-2	8

(III)
x	-1	1	4	5
y	31	6	0	2

(IV)
x	0	8	2	4
y	22	0	0	-14

(a) Find the Lagrange form of the interpolating polynomial for each data set.

(b) Find the Newton form of the interpolating polynomial for each data set.

(c) Verify that your answers to (a) and (b) simplify to the same polynomial.

(d) Add the point $(3, 1)$ to each data table above, and find the interpolation polynomial for the modified data sets.

Problem 2. Consider the following data table:

x	0	2	3
y	13	9	-20

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.

(c) Verify that your answers to (a) and (b) simplify to the same polynomial.

(d) Suppose that this data was given by the function $f(x) = 1 - x^2 + 24\sin(\frac{\pi}{3}x + \frac{\pi}{6})$. Find a (tight) upper bound for the interpolation error.

Problem 3. Consider the following data table:

x	-1	0	1
y	12	6	4

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.

(c) Suppose that this data was given by the function $f(x) = \dfrac{12}{2+x}$. Find a (tight) upper bound for the interpolation error.

Problem 4. Consider the following data table:

x	0	1	4
y	0	1	64

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.

(c) Suppose that this data was given by the function $f(x) = x^3$. Find a (tight) upper bound for the interpolation error.

Problem 5. Suppose we have $n+1$ nodes $x_0, \ldots, x_n$. Let L be the transformation defined as follows:

$$L(f) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$$

where $\ell_i(x)$ are the Lagrange cardinal functions. So L takes a function as input and gives the interpolation polynomial of degree n as output.

(a) Prove that L is a linear transformation: $L(af + bg) = aL(f) + bL(g)$.

(b) Prove that, if q is a polynomial of degree n, then $L(q) = q$.

(c) Prove that $\displaystyle\sum_{i=0}^{n} \ell_i(x) = 1$ for all $x \in \mathbb{R}$. (Hint: Try interpolating $f(x) = 1$.)

(d) Suppose p is a polynomial of degree $\leq n$ that interpolates f at the given $n+1$ nodes. Prove that

$$f(x) - p(x) = \sum_{i=0}^{n} [f(x) - f(x_i)]\ell_i(x)$$

Problem 6. In each part below, you are given a function and a set of nodes. Find the interpolation error for the polynomial of least degree that interpolates each function at these nodes.

(a) $f(x) = \dfrac{1}{1+25x^2}$, $\quad x_0 = 0$ and $x_1 = 1$

(b) $f(x) = \cos(2\pi x)$, $\quad x_0 = -3$, $x_1 = 0$, $x_2 = 3$

(c) $f(x) = e^{-x}$, $\quad$ any 20 nodes in $[0, 1]$

(d) $f(x) = \ln(x+1)$, $\quad$ any 10 nodes in $[0, 2]$

Problem 7. Let $f \in C^{n+1}([a, b])$, and suppose p is the polynomial of degree at most n that interpolates f at the $n+1$ distinct nodes $x_0, \ldots, x_n$.

(a) Let $t \in [a, b]$ be different from the $n+1$ nodes above. Derive the polynomial q that interpolates f at the $n+2$ distinct nodes $x_0, \ldots, x_n, t$. (Hint: the formula for $q(x)$ should involve $p(x)$.)

(b) Using part (a), show that

$$f(t) - p(t) = f[x_0, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

(c) Prove that there exists $\xi \in (a, b)$ such that

$$f[x_0, \ldots, x_n, t] = \frac{1}{(n+1)!} f^{(n+1)}(\xi)$$

Problem 8. Let f be a given function and let $h > 0$. Prove that

$$f(x+2h) - 2f(x+h) + f(x) = h^2 f''(\xi)$$

for some $\xi \in (x, x+2h)$. (Hint: consider interpolating f at the points x, $x+h$, and $x+2h$.)

Problem 9. Let f be a given function and let $x_0, \ldots, x_n$ be distinct nodes. Suppose u is a function that interpolates f at $x_0, \ldots, x_{n-1}$ and v is a function that interpolates f at $x_1, \ldots, x_n$. Prove that the function

$$\phi(x) = \frac{(x_n - x)u(x) + (x - x_0)v(x)}{x_n - x_0}$$

interpolates f at all the nodes $x_0, \ldots, x_n$.

Problem 10. Consider the following data tables:

(I)

x	1	4	5
$f(x)$	27	0	-1
$f'(x)$	0	9	-3

(II)

x	-1	0	2	7
$f(x)$	-12	24	0	18
$f'(x)$		10		

(III)

x	0	1	2
$f(x)$	4	-1	12
$f'(x)$	-2	3	
$f''(x)$		8	

(IV)

x	a	b	c
$f(x)$	$f(a)$	$f(b)$	$f(c)$
$f'(x)$	$f'(a)$	$f'(b)$	$f'(c)$
$f''(x)$	$f''(a)$	$f''(b)$	$f''(c)$
$f'''(x)$	$f'''(a)$	$f'''(b)$	$f'''(c)$

(a) Find the Hermite interpolating polynomial for the data sets (I), (II), and (III).

(b) For data set (IV), compute the divided-difference table (you can leave the values as divided-differences, i.e. $f[x_i, x_j]$). Substitute derivative values where appropriate.

Problem 11. Consider the following data table:

x	x_0	x_1
$f(x)$	a_0	a_1
$f'(x)$	b_0	

(a) Find the Hermite interpolating polynomial $p(x)$ for this data.

(b) Compute $p(x_0)$, $p(x_1)$, and $p'(x_0)$. Explain why your answers make sense.

Problem 12. Let $x_0, \ldots, x_n$ be nodes, not necessarily distinct. Prove that a polynomial interpolates 0 at these nodes if and only if it contains the factor $\prod_{j=0}^{n}(x - x_j)$.

Problem 13. Let f be a given function and let $x_0, \ldots, x_n$ be nodes, not necessarily distinct. Suppose g is a function that interpolates f at these nodes, and h interpolates 0 at these nodes. Prove that $g \pm ch$ interpolates f at these nodes, for some number c.

Problem 14. Let $t_0 < t_1 < t_2 < t_3 < t_4$ be knots, and let $S(x)$ be the spline function that interpolates these knots, where

$$S(x) = \begin{cases} S_0(x), & x \in [t_0, t_1] \\ S_1(x), & x \in [t_1, t_2] \\ S_2(x), & x \in [t_2, t_3] \\ S_3(x), & x \in [t_3, t_4] \end{cases}$$

(a) What conditions does S need to satisfy in order to be a *linear spline*?

(b) What conditions does S need to satisfy in order to be a *quadratic spline*?

(c) What conditions does S need to satisfy in order to be a *cubic spline* (not necessarily natural)?

(d) What conditions does S need to satisfy in order to be a *natural cubic spline*?

Problem 15. Consider the function

$$S(x) = \begin{cases} 1 - 4x + x^3, & x \in [0, 2] \\ a + b(x-2) + c(x-2)^2 + d(x-2)^3, & x \in [2, 5] \end{cases}$$

Find a, b, c, d so that $S(x)$ is a natural cubic spline on $[0, 5]$.

Problem 16. Determine whether the following function is a cubic spline:

$$f(x) = \begin{cases} 2(x+1) + (x+1)^3, & x \in [-1, 0] \\ 3 + 5x + 3x^2, & x \in [0, 1] \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3, & x \in [1, 2] \end{cases}$$

Problem 17. Let $f \in C^2([a, b])$ and let $a = t_0 < t_1 < \cdots < t_n = b$ be knots in $[a, b]$. Suppose $S(x)$ is the *natural* cubic spline that interpolates f at these knots $\{t_i\}_{i=0}^n$.

(a) Define $g(x) := f(x) - S(x)$. Show that $\displaystyle\int_a^b S''(x)g''(x)\,dx \geq 0$. (Hint: use integration by parts.)

(b) Using part (a), prove:

$$\int_a^b [S''(x)]^2\,dx \leq \int_a^b [f''(x)]^2\,dx$$

Problem 18. Consider the knots $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$. Find the natural cubic spline function on these knots that interpolates the data points $(-1, 13)$, $(0, 7)$, and $(1, 9)$.

Problem 19. Consider the function

$$S(x) = \begin{cases} x^3, & x \in [0, 1] \\ \frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c, & x \in [1, 3] \end{cases}$$

Find a, b, c so that $S(x)$ is a cubic spline on $[0, 3]$. Is $S(x)$ a natural cubic spline?

Problem 20. Let $\{t_i\}$ be a collection of knots, where $i \in \mathbb{Z}$, and let $k \geq 1$. Suppose $x \notin (t_i, t_{i+k+1})$, for some index i. Prove that $B_i^k(x) = 0$.

Problem 21. Consider the collection of knots $\{t_i\}$ given by $t_i = i$, where $i \in \mathbb{Z}$. Prove that $B_0^k(x) = B_i^k(x + t_i)$.

Problem 22. Consider the collection of knots $\{t_i\}$ given by $t_i = i$, where $i \in \mathbb{Z}$. Define $h_{ij} = t_i - t_j$. Prove that $B_i^k(x) = B_j^k(x - h_{ij})$.

Problem 23. Prove that if $\displaystyle\sum_{n=-\infty}^{\infty} c_i B_i^k(x) = 0$ for all x, then $c_i = 0$ for all i.

Problem 24. In each part below, you are given a set of knots, a set of nodes, and a set of B-splines. Determine if it is possible to perform interpolation at the given nodes using linear combinations of the given B-splines.

(a) knots: $t_i = i$, where $i \in \mathbb{Z}$
nodes: $x_0 = 1.2$, $x_1 = 2.4$, $x_2 = 3.1$
B-Splines: B_0^1, B_1^1, B_2^1

(b) knots: $t_i = i$, where $i \in \mathbb{Z}$
nodes: $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.8$, $x_3 = 8.5$
B-Splines: B_0^3, B_1^3, B_2^3, B_3^3

(c) knots: $t_i = 2i$, where $i \in \mathbb{Z}$
nodes: $x_1 = 2.1$, $x_2 = 4.3$, $x_3 = 4.7$
B-Splines: B_1^1, B_2^1, B_3^1

(d) knots: $t_i = i/2$, where $i \in \mathbb{Z}$
nodes: $x_1 = 0.75$, $x_2 = 2$, $x_3 = 2.9$, $x_4 = 3.1$, $x_5 = 3.7$
B-Splines: B_1^2, B_2^2, B_3^2, B_4^2, B_5^2