Math 170B, Spring 2018
Review for Midterm 1

Problem 1. Let $f(x) = e^x$.

(a) Find the Taylor Series for f about the point $x = 0$. Include two or three versions of the remainder term.

(b) Find the minimum number of terms needed to compute e with an error less than 10^{-8}.

(c) Find the minimum number of terms needed to compute e^2 with an error less than 2^{-m}. Your answer should be an inequality.

(d) Suppose we use the Taylor polynomial of degree $n = 2$ to compute e. Determine the error bound given by this approximation.

Problem 2. Find the order of convergence for each of the following sequences.

(a) $x_n = 3^{-n} + 1$

(b) $x_{n+1} = \frac{1}{2}x_n$, with $x_0 \in \mathbb{R}$

(c) $x_{n+1} = \frac{1}{2}x_n^p$, with $x_0 \in \mathbb{R}$ and $p \geq 1$

(d) $x_{n+1} = \frac{1}{2}(x_n^2 + 1)$, with $x_0 \in (0, 1)$.

(e) $x_n = \dfrac{1}{n!}$

(f) $x_{n+1} = x_n\left(3 - \dfrac{x_n}{a}\right)$, where $a > 0$.

(g) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$, where $a > 0$.

Problem 3. Prove the following statements.

(a) $5n^3 + 9n^2 + 1 = \mathcal{O}(n^3)$

(b) $5n^3 + 9n^2 + 1 = \mathcal{O}(n^4)$

(c) $5n^3 + 9n^2 + 1 \neq \mathcal{O}(n^3)$

(d) $\frac{1}{n^2} = o(\frac{1}{n})$

(e) $\frac{1}{n^2} \neq o(\frac{1}{n^2})$

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable at a point $x \in \mathbb{R}$, and let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$. Prove

$$f'(x) = \lim_{h \to \infty} \frac{f(x + \alpha h) - f(x + \beta h)}{\alpha h - \beta h}$$

Problem 5. Let $\{x_n\}$, $\{y_n\}$ and $\{\alpha_n\}$ be sequences in $\mathbb{R}$. Prove the following statements.

(a) $x_n = x + o(1)$ if and only if $\lim_{n \to \infty} x_n = x$.

(b) If $x_n = \mathcal{O}(\alpha_n)$, then $cx_n = \mathcal{O}(\alpha_n)$, where c is a constant.

(c) If $x_n = \mathcal{O}(\alpha_n)$ and $y_n = \mathcal{O}(\alpha_n)$, then $x_n + y_n = \mathcal{O}(\alpha_n)$.

(d) If $x_n = o(\alpha_n)$ and $y_n = o(\alpha_n)$, then $x_n + y_n = o(\alpha_n)$.

(e) If $x_n = \mathcal{O}(\alpha_n)$, $y_n = \mathcal{O}(\alpha_n)$, and $\alpha_n \to 0$, then $x_n y_n = o(\alpha_n)$.

Problem 6. Let F be a C^1 function, and suppose the sequence $\{x_n\}$ is defined by $x_{n+1} = F(x_n)$. Assume $\lim_{c \to \infty} x_n = x \in \mathbb{R}$. Use the Mean Value Theorem to show that $x_{n+2} - x_{n+1} = o(x_{n+1} - x_n)$.

Problem 7. Consider the equation $(x^3 - 1)y + e^x y^2 + \cos(x) - 1 = 0$. Show that at the point $(0, 0)$, there exists a continuously differentiable function f such that $y = f(x)$ in a neighborhood of $(0, 0)$.

Problem 8. Let $[a, b]$ be an interval in $\mathbb{R}$. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that $f(a)$ and $f(b)$ have different signs, and let r be a zero of f, i.e. $f(r) = 0$. Prove that the error e_n in approximating r with the Bisection Method is bounded as follows:

$$|e_n| \leq 2^{-(n+1)}(b - a)$$

Problem 9. Let $c_n = \frac{1}{2}(a_n + b_n)$, $r = \lim_{n \to \infty} c_n$, and $e_n = r - c_n$. Suppose that after running the Bisection method, we have a sequence of intervals $[a_n, b_n]$, with given initial interval $[a_0, b_0]$.

(a) Prove that $e_n = \mathcal{O}(2^{-n})$ as $n \to \infty$.

(b) Prove that $e_n \geq e_{n+1}$ for all $n \geq 0$.

(c) Prove that $|c_n - c_{n+1}| = 2^{-(n+2)(b_0 - a_0)}$.

Problem 10. Consider running the Bisection method with the starting interval $[1, 4]$. Determine the number of steps n that need to be taken to guarantee the error bound $|r - c_n| \leq 10^{-8}$.

Problem 11. Consider this problem: we are trying to approximate the square root of 3.

(a) Formulate this problem as a root-finding problem.

(b) Find two numbers a, b such that $a^2 < 3 < b^2$. Then, considering the interval $[a, b]$, find the error given by the Bisection method after n steps.

(c) Derive the Newton iteration for this problem. Give an estimate of the error e_{n+1}.

(d) Derive the Secant iteration for this problem. Give an estimate of the error e_{n+1}.

(e) Using parts (b)-(d), use the Bisection method to compute c_2 and use Newton's Method to compute x_2 given $x_0 = 1$. Using the x_0 and x_1 from Newton's method, compute x_2 using the Secant method.

Problem 12. Devise a Newton iteration to compute the nth root of a positive number α.

Problem 13. Perform two iterations of Newton's method on the following problems.

(a) $x^2 - 5x + 6 = 0$, with $x_0 = 1$

(b) $4 - x^4 = x^3$, with $x_0 = 1$

(c) $e^{-x} - x = 0$, with $x_0 = 0$

(d) $\begin{cases} 2x_1^2 - x_1 x_2 = 1 \\ x_1 + x_2^3 = 0 \end{cases}$, with $x^{(0)} = (1, 0)^T$

(e) $\begin{cases} 5x_1 - x_1^2 + 2x_2 = 10 \\ (x_1 - 2)^3 + (x_2 - 1)^2 = 4 \end{cases}$, with $x^{(0)} = (2, 2)^T$

Problem 14. Perform two iterations of the Secant method on the following problems.

(a) $x^2 - 5x + 6 = 0$, with $x_0 = 0, x_1 = 1$

(b) $4 - x^4 = x^3$, with $x_0 = 1, x_1 = 2$

(c) $e^{-x} - x = 0$, with $x_0 = 0, x_1 = 1$

Problem 15. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a given function, and let $f(r) = 0$ for some $r \in \mathbb{R}$. Recall that the symbol $\approx$ means "approximately", e.g. if we write $a \approx b$ this means the value of a is approximately equal to the value of b.

(a) Suppose we are given an interval $[a, b]$ containing r. Write down the error bound for the Bisection Method at the nth step of the algorithm. Then, denoting the error by e_n, show that $|e_n| \approx \frac{1}{2}|e_{n-1}|$.

(b) Now, given an approximation x_n to the root r, derive the error term e_{n+1} for Newton's method. Using this error term and a first-order Taylor expansion with remainder, show that $|e_{n+1}| \approx C|e_n|^2$, where C is a positive constant.

(c) Now, given two approximations x_n and x_{n-1} to the root r, derive the error term e_{n+1} for the secant method. Use this error term to show that $|e_{n+1}| \approx A|e_n|^{(1+\sqrt{5})/2}$, where A is a positive constant.

Problem 16. Prove that the following functions have a fixed point in the indicated interval.

(a) $F(x) = \frac{1}{2}x^2 + 3$ in $[0, \frac{1}{4}]$

(b) $F(x) = e^{-x} - x$ in $[0, 1]$

(c) $F(x) = \sin(x) - x$ in $[0, 1]$

Problem 17. Consider the fixed-point problem $F(x) = x$. Devise a Newton iteration for this problem.

Problem 18. Given an interval $[a, b]$, let $F \in C^1([a, b])$ and suppose $|F'(x)| \leq 1$ on $[a, b]$. Prove that F has a fixed point in $[a, b]$.