Math 170B, Spring 2018
Review for Final

Problem 1.

(a) Derive an approximation to the first derivative $f'(x)$ using the points x, $x+h$, and $x+2h$. Give the exact error term for this approximation.

(b) Derive an approximation to the first derivative $f'(x)$ using the points $x+h$ and $x-h$. Give the exact error term for this approximation.

(c) Derive an approximation to the second derivative $f''(x)$ using the points x, $x+h$, and $x+2h$. Give the exact error term for this approximation.

(d) Derive an approximation to the second derivative $f''(x)$ using the points $x-h$, x, and $x+h$. Give the exact error term for this approximation.

Problem 2.

(a) Derive the finite difference $f'(x) \approx \dfrac{f(x+2h) - f(x-h)}{3h}$

(b) Using Taylor series, write out the first few terms of the error expansion.

(c) Perform one step of Richardson extrapolation to derive a numerical method of higher order. What is the order of the new method?

Problem 3. Suppose we are trying to approximate a quantity L using a numerical method $\phi(h)$. Suppose further that L is given exactly by $L = \phi(h) + \sum_{j=1}^{\infty} a_j h^j$, where $a_j \in \mathbb{R}$.

(a) Based on the given formula for L, what is the order of the numerical method $\phi(h)$?

(b) Perform two steps of Richardson Extrapolation to derive a numerical method of higher order. What is the order of the new method?

Problem 4. Using combinations of n and n^2, perform two steps of Richardson Extrapolation on $L = x_n + a_1 n^{-1} + a_2 n^{-2} + a_3 n^{-3} + \ldots$

Problem 5.

(a) Give the recursive definition for $T_n(x)$, the Chebyshev polynomials of the first kind.

(b) Show that $\cos((n+1)\theta) = 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta)$. Hint: use the the identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$.

(c) Use part (b) to prove that $T_n(x) = \cos(n\cos^{-1}(x))$ for all $n \geq 0$.

Problem 6. Prove that $\displaystyle\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = 0$ unless $n = m$. Hint: use the trig substitution $x = \cos(\theta)$. (Remark: This shows that the Chebyshev polynomials of the first kind are an *orthogonal* family of polynomials.)

Problem 7.

(a) Give the closed-form formula for computing $U_{n+1}(x)$, the Chebyshev polynomials of the second kind.

(b) Prove that these polynomials are generated recursively by $\begin{cases} U_0(x) = 1, \quad U_1(x) = 2x \\ U_{n+1} = 2xU_n - U_{n-1} \end{cases}$ for $n \geq 1$.

Problem 8. Let T_n be the Chebyshev polynomials of the first kind, and let U_n be the Chebyshev polynomials of the second kind. Prove that $T_n' = nU_{n-1}$.

Problem 9. Find the interpolation polynomial $\lambda(x)$ such that $\lambda(c) = a$ and $\lambda(d) = b$. (This is the change of variables formula to translate $[c,d]$ to $[a,b]$.)

Problem 10. Consider the integral $\int_0^1 f(x)\,dx$. For each set of nodes below, derive a quadrature rule to approximate this integral. You should derive each formula twice: once using Lagrange interpolation, and once using the method of undetermined coefficients.

(a) $\{\frac{1}{3}, \frac{2}{3}\}$, exact for all polynomials of degree ≤ 2

(b) $\{0, \frac{1}{2}, 1\}$, exact for all polynomials of degree ≤ 3

(c) $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, exact for all polynomials of degree ≤ 3

(d) Transform each formula above into one for integration over $[a,b]$.

Problem 11. Determine the values of A, B, C that make the formula

$$\int_0^2 xf(x)\,dx \approx Af(0) + Bf(1) + Cf(2)$$

exact for all polynomials of degree as high as possible. What is the maximum degree?

Problem 12. We will examine the *Midpoint Rule* in this problem.

(a) Derive the approximation $\int_{-1}^{1} f(x)\,dx \approx Af(\alpha)$ that is exact for all linear functions.

(b) Transform the formula in part (a) to one for integration over $[a, b]$.

(c) Transform the formula in part (a) to one for integration over $[x_{i-1}, x_{i+1}]$.

(d) Using part (c), derive the *Composite Midpoint Rule* for approximating $\int_{x_0}^{x_n} f(x)\,dx$. (Assume the nodes $x_0, \ldots, x_n$ are given.)

Problem 13. We will examine the *Trapezoid Rule* in this problem. Suppose the nodes $x_0, \ldots, x_n$ are given.

(a) Using the nodes x_{i-1} and x_i, derive an approximation to $\int_{x_{i-1}}^{x_i} f(x)\,dx$ that is exact for all polynomials of degree ≤ 1.

(b) Determine the error for the quadrature rule in part (a) by integrating the error term for polynomial that interpolates $f(x)$ at x_{i-1} and x_i.

(c) Using part (a), derive the *Composite Trapezoid Rule* for approximating $\int_{x_0}^{x_n} f(x)\,dx$.

(d) Using part (b), derive the error term for the quadrature rule in part (c).

(e) Suppose we use the Composite Trapezoid Rule to approximate $\int_a^b f(x)\,dx$, where $|f''(x)| \leq M$. What is the minimum number of subintervals needed to have the absolute error $< \epsilon$.

Problem 14. Let $f(x)$ be a given function and let $x_0 < x_1 < x_2 < x_3$ be given nodes.

(a) Find the Lagrange form of the polynomial interpolating $f(x)$ at x_1 and x_2.

(b) Derive a quadrature rule to approximate $\int_{x_0}^{x_3} f(x)\,dx$ that uses the nodes x_1 and x_2.

Problem 15. Consider the quadrature rule $\sum_{i=1}^{n} A_i f(x_i)$. In this problem, we will examine *Gaussian quadrature* formulas. For Gaussian quadrature, both the coefficients A_i and the nodes x_i need to be determined (this yields methods of higher order than standard Newton-Cotes formulas).

(a) Determine A and x_0 so that the approximation $\int_0^1 f(x)\sqrt{x}\,dx \approx Af(x_0)$ is exact for all linear polynomials.

(b) Determine w and α so that the approximation $\int_{-1}^1 f(x)\,dx \approx wf(-\alpha) + wf(\alpha)$ is exact for all polynomials of degree ≤ 3.

Problem 16. Suppose $x_0, \ldots, x_n$ are nodes in $[0, 1]$. For a given function f, let p be the degree n polynomial that interpolates f at these nodes. Assume $|f^{(n+1)}(x)| \leq M$ on $[0, 1]$, where $M \in \mathbb{R}$.

(a) If no other information is given about the location of the nodes, what is an upper bound for the error $|\int_0^1 f(x)\,dx - \int_0^1 p(x)\,dx|$?

(b) Explain how to choose the $n+1$ nodes so that the approximation is optimized (i.e. the error is minimized).

(c) If we use the nodes in part (b), what is the best upper bound that can be achieved?

Problem 17. Find the order of convergence for each of the following sequences.

(a) $x_n = 4^{-n} + 1$

(b) $x_{n+1} = \frac{1}{3}x_n^p$, where $x_0 \in \mathbb{R}$ and $p \geq 1$

(c) $x_{n+1} = \frac{1}{2}(x_n^2 + 1)$, where $x_0 \in (0, 1)$

(d) $x_{n+1} = \dfrac{x_n(x_n^2 + 3R)}{3x_n^2 + R}$, where $x_0 > 1$ and $R > 1$

(e) $x_n = 2^{-n^2}$

(f) $x_n = 3^{-2^n}$

(g) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$, where $a > 0$ and $x_0 > 0$.

Problem 18. For each of the intervals below, suppose the Bisection Method is used to approximate some root r. Determine the number of steps n that need to be taken to guarantee the error bound $|r - c_n| \leq \epsilon$, where ϵ is specified in each part.

(a) $[0, 4]$, $\epsilon = 10^{-10}$

(b) $[2, 3]$, $\epsilon = 10^{-6}$

(c) $[1, 2]$, $\epsilon > 0$ (arbitrary)

Problem 19. Formulate each problem below as a root-finding problem, and then give the Newton iteration for each problem.

(a) Find the square root of 7.

(b) Find the nth root of a positive number R.

(c) Find the multiplicative inverse of a number α.

(d) Find a fixed-point of the function $F\colon \mathbb{R} \to \mathbb{R}$.

(e) Find the minimum of a convex function $f\colon \mathbb{R} \to \mathbb{R}$.

Problem 20. Perform two iterations of Newton's method on the following problems.

(a) $x^2 - 7x + 10 = 0$, with $x_0 = 1$

(b) $4 - 2x^3 = x^2$, with $x_0 = 1$

(c) $e^{-x} - x = 0$, with $x_0 = 0$

(d) $\begin{cases} 2x_1^2 - x_1 x_2 = 1 \\ x_1 + x_2^3 = 0 \end{cases}$, with $x^{(0)} = (1, 0)^T$

(e) $\begin{cases} 5x_1 - x_1^2 + 2x_2 = 10 \\ (x_1 - 2)^3 + (x_2 - 1)^2 = 4 \end{cases}$, with $x^{(0)} = (2, 2)^T$

Problem 21. Perform two iterations of the Secant method on the following problems.

(a) $x^2 - 7x + 10 = 0$, with $x_0 = 0, x_1 = 1$

(b) $4 - 2x^3 = x^2$, with $x_0 = 1, x_1 = 2$

(c) $e^{-x} - x = 0$, with $x_0 = 0, x_1 = 1$

Problem 22. Prove that the following functions have a fixed point in the indicated interval.

(a) $F(x) = \frac{1}{2}e^{-x}$ in $[0, 2]$

(b) $F(x) = \dfrac{1}{\sqrt{3+x^2}}$ in $[0, 1]$

(c) $F(x) = \cos(x)$ in $[0, 1]$

Problem 23. Consider the following data tables:

(I)

x	0	1	2
y	10	1	4

(II)

x	-1	0	1
y	-13	4	-1

(III)

x	0	1	2
y	11	-2	3

(a) Find the Lagrange form of the interpolating polynomial for each data set.

(b) Find the Newton form of the interpolating polynomial for each data set.

(c) Verify that your answers to (a) and (b) simplify to the same polynomial.

(d) Add the point $(7, 15)$ to each data table above, and find the interpolation polynomial for the modified data sets.

Problem 24. Consider the following data table:

x	0	2	3
y	13	9	-20

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.

(c) Verify that your answers to (a) and (b) simplify to the same polynomial.

(d) Suppose that this data was given by the function $f(x) = 1 - x^2 + 24\sin(\frac{\pi}{3}x + \frac{\pi}{6})$. Find a (tight) upper bound for the interpolation error.

Problem 25. Consider the following data table:

x	0	2	3
y	0	6	24

(a) Find the Lagrange form of the interpolating polynomial for this data.

(b) Find the Newton form of the interpolating polynomial for this data.

(c) Suppose that this data was given by the function $f(x) = x^3 - x$. Find a (tight) upper bound for the interpolation error.

Problem 26. Let $f \in C^{n+1}([a, b])$, and suppose p is the polynomial of degree at most n that interpolates f at the $n + 1$ distinct nodes $x_0, \ldots, x_n$.

(a) Let $t \in [a, b]$ be different from the $n + 1$ nodes above. Derive the polynomial q that interpolates f at the $n + 2$ distinct nodes $x_0, \ldots, x_n, t$. (Hint: the formula for $q(x)$ should involve $p(x)$.)

(b) Using part (a), show that

$$f(t) - p(t) = f[x_0, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

(c) Prove that there exists $\xi \in (a, b)$ such that

$$f[x_0, \ldots, x_n, t] = \frac{1}{(n+1)!} f^{(n+1)}(\xi)$$

Problem 27. In each part below, you are given a function and a set of nodes. Find the interpolation error for the polynomial of least degree that interpolates each function at these nodes.

(a) $f(x) = \dfrac{1}{1 + 25x^2}$, any 20 nodes in $[-1, 1]$

(b) $f(x) = \cos(2\pi x)$, $x_0 = -3$, $x_1 = 0$, $x_2 = 3$

(c) $f(x) = e^{-x}$, any 10 nodes in $[-1, 1]$

(d) $f(x) = \ln(x + 2)$, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$

Problem 28. Suppose each function below is interpolated using the roots of the Chebyshev polynomials of the first kind. Determine an optimal upper bound on the interpolation error.

(a) $f(x) = \dfrac{1}{1+25x^2}$, 20 Chebyshev nodes in $[-1, 1]$

(b) $f(x) = e^{-x}$, 10 Chebyshev nodes in $[-1, 1]$

(c) $f(x) = \ln(x+2)$, 17 Chebyshev nodes in $[-1, 1]$

Problem 29. Find the Hermite interpolating polynomial for each data set below:

(I)

x	1	4	5
$f(x)$	27	0	-1
$f'(x)$	0	9	-3

(II)

x	0	1	2
$f(x)$	4	-3	8
$f'(x)$		1	-2
$f''(x)$		10	

Problem 30. Consider the function

$$S(x) = \begin{cases} 1 + a(x-1) + b(x-1)^3, & x \in [1, 2] \\ 1 + c(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & x \in [2, 3] \end{cases}$$

Find a, b, c, d so that $S(x)$ is a natural cubic spline on $[1, 3]$ that interpolates the data $(1, 1)$, $(2, 1)$ and $(3, 0)$.

Problem 31. Determine whether the following function is a natural cubic spline:

$$f(x) = \begin{cases} 1 + x - x^2, & x \in [0, 1] \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3, & x \in [1, 2] \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3, & x \in [2, 3] \end{cases}$$

Problem 32. Let $\{t_i\}$ and $\{u_i\}$ be knot sequences, where $u_i = t_i + h$, for all $i \in \mathbb{Z}$. Let Q_j^k be the B-splines associated to the knots t_i, and let R_j^k be the B-splines associated to the knots u_i. Prove that $Q_i^k(x - h) = R_i^k(x)$.

Problem 33. In each part below, you are given a set of knots, a set of nodes, and a set of B-splines. Determine if it is possible to perform interpolation at the given nodes using linear combinations of the given B-splines.

(a)
knots: $t_i = 2i$, where $i \in \mathbb{Z}$
nodes: $x_1 = 2.1$, $x_2 = 4.3$, $x_3 = 4.7$
B-Splines: B_1^1, B_2^1, B_3^1

(b)
knots: $t_i = i/2$, where $i \in \mathbb{Z}$
nodes: $x_1 = 0.75$, $x_2 = 2$, $x_3 = 2.9$, $x_4 = 3.1$, $x_5 = 3.7$
B-Splines: B_1^2, B_2^2, B_3^2, B_4^2, B_5^2

Problem 34. Let $\{t_i\}$ be the knot sequence given by $t_i = i \in \mathbb{Z}$. Consider the B-splines B_1^1, B_2^1, B_3^1, B_4^1, B_5^1, and the nodes $\{1.1, 2.1, 3.4, 5.1, 5.5\}$.

(a) Show that these B-splines can be used to interpolate the nodes.

(b) Using B_1^1 and B_2^1, find a formula for the linear spline, defined on $[1, 4]$ that interpolates $(1.1, 2)$ and $(2.1, 4)$.

(c) Using B_1^1, B_2^1, and B_3^1, find a formula for the linear spline, defined on $[1, 5]$ that interpolates $(1.1, 2)$, $(2.1, 4)$, and $(3.4, -1)$.