Math 140C Proof Writing Standards

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Abstract

This document outlines certain standards that will be used for evaluating your work in this course. These are provided in an effort to help you understand what constitutes a *correct, complete*, *concise* proof that will earn full credit in homework write-ups and exams.

Neglecting to observe any of these standards in your writing is grounds for loss of credit. While some of these rules may seem "stylistic" (and are meant to guide your professional mathematical writing), many are truly necessary for accurate, logically tight proofs.

Quoting Results

Probably the most important skill in writing proofs well at this level is meticulously verifying the hypotheses of results that are being used. To that end, the following guidelines are among the most essential.

1. Whenever you use a theorem or proposition in your writing, you should clearly state *all* hypotheses, and you should state *all* the conclusions that you are going to use.
2. Hypotheses of every result must be explicitly verified in your writing. If they are trivially fulfilled, you can merely acknowledge them, but each must be individually mentioned.
3. If you want to use the result of a homework problem, it must be one that was already assigned and completed, or you must work it out as a lemma.
4. Many times, it's tempting to use a result implicitly! In particular, any statement of the form "therefore...," "it suffices to show that...," or "without loss of generality...," must be specifically justified. ("It suffices" and "WLOG" are the most sneaky in this sense).
5. If you want to use a claim that has not been proven in the textbook, you must prove it by hand as a lemma. If you want to use another book/webpage as a source, you must include the proof and acknowledge the source (e.g., "following the proof of `<statement>` in `<source>`, we can see ...," with proper attribution).
6. To use a claim proven in class, but not the textbook, treat it the same as an outside source. (Most of what is proved in lecture is found in the text).

Remark These guidelines will be felt in verifying definitions in the upcoming material on Lebesgue measure and integration. For example, to use a result concerning a σ-ring, there are a number of conditions in the definition of σ-ring that are necessary to check first (and they *must* be checked!).

Inequalities

To prove an inequality, every inequality in your string must be individually justified. This may be omitted if the justification is obvious (adding a positive term, taking the absolute value, etc), but if it requires anything

more complex than, say, the triangle inequality, the justification ought to be explicitly stated. A (contrived) example:

$$\begin{aligned}
\mu(\cup_{i=1}^N A_i) + 2xy + \sum_{i=1}^n |t_i| &\leq \mu(\cup_{i=1}^N A_i) + x^2 + y^2 + \sum_{i=1}^n |t_i| && \text{(as } x^2 - 2xy + y^2 = (x-y)^2 \geq 0\text{)} \\
&\leq \mu(\cup_{i=1}^N A_i) + x^2 + y^2 + \sqrt{n \sum_{i=1}^n |t_i|^2} && \text{(Cauchy-Schwarz with } u_i = 1, v_i = |t_i|\text{)} \\
&\leq \sum_{i=1}^N \mu(A_i) + x^2 + y^2 + \sqrt{n \sum_{i=1}^n |t_i|^2} && \text{(additivity, non-negativity of } \mu\text{)}
\end{aligned}$$

Calculations

1. When something must be explicitly calculated, show sufficient detail such that your reader can fill in the blanks only with their eyes. In particular, if some function has been defined (or an identity has been proven) elsewhere on the page, it is best practice to apply that definition/identity in your writing explicitly. If $f(x) = x^2$,

   $$f(x) - f(-x) = x^2 - (-x)^2 = 0$$

   is much better than

   $$f(x) - f(-x) = 0.$$

   In the second case, it is unclear whether you, as a student preparing a proof, have actually verified the conclusion you want to use, and as such, materially diminishes the quality of your solution.

2. *Calculations without context are bad.* All of your work must be written in complete English sentences, and no sentence is allowed to start with symbolic math. For example,

   $$\text{We recall that } \cos^2\theta + \sin^2\theta = 1\text{, and this completes the proof.}$$

   is good, but

   $$\cos^2\theta + \sin^2\theta = 1 \text{ QED.}$$

   is not.

   $$\cos^2\theta + \sin^2\theta = 1\text{, and the desired conclusion follows}$$

   would be correct, but of decidedly bad form.

Assumptions

1. Assumptions should be specified before they are used. For example,

   $$\text{If } f \in C^2\text{, corollary 9.42 gives that } D_{12}f - D_{21}f = D_{21}f - D_{21}f = 0$$

   is far better than

   $$D_{12}f - D_{21}f = D_{21}f - D_{21}f = 0\text{, so the equation holds for } f \in C^2 \text{ by corollary 9.42.}$$

   By the time your reader approaches a statement, it should already be true (and justified), instead of being made true (or clear) by a later line.

   Remark This has been missed a lot when a calculation needs to be restricted away from a point or points where a function is undefined, and then declaring the restriction later; in particular, protect yourself from dividing by 0 before the fact, not after.

2. When you want to do a proof by contradiction, you should use a phrase such as "Assume towards contradiction that `<assumption>`" or "To draw a contradiction, assume `<assumption>`," etc. This again allows the reader to enjoy completely resolved logic as they go through your writing. (In addition, you should also check if you can avoid the contradiction by proving the contrapositive of the statement

at hand – unnecessary contradiction proofs are informally considered somewhat gauche, even if they are correct).

Existence and Well-Definedness

1. Whenever you say "Let _____ be such that...," you must substantiate the claim that such an object exists. This may often feel silly, but it is absolutely necessary – consider the following proof:

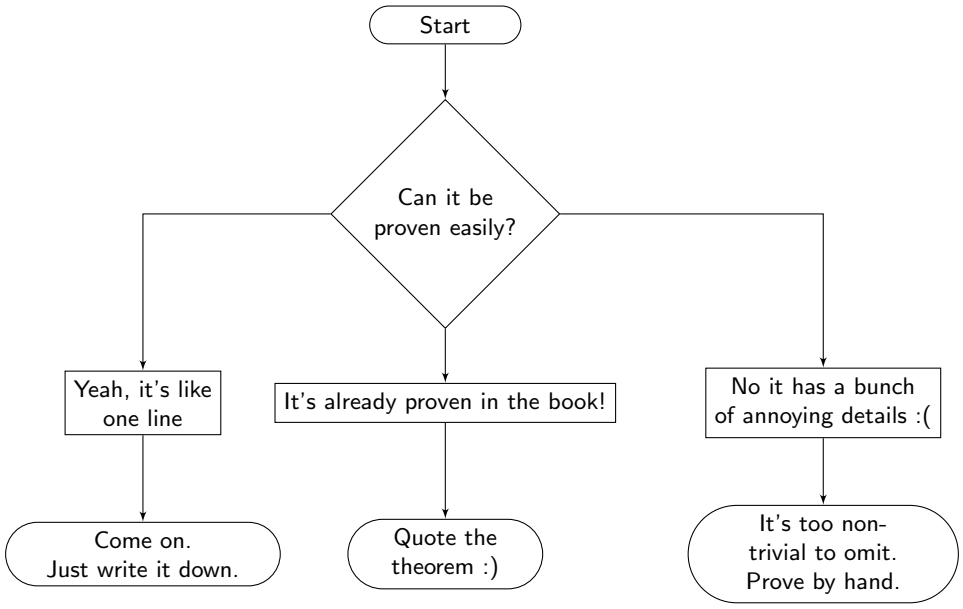
   Let $f : \mathbb{N} \to \mathbb{R}$ be a bijection. Then $\mathbb{R}$ is countable. QED.

   The existence and well-definedness of many mathematical objects are often non-trivial results, and as such must be rigorously established. An immediate example is the claim that every vector space has a basis – this result is actually equivalent to the Axiom of Choice when you include vector spaces of infinite dimension.

2. In support of the above, whenever you introduce a function, its domain must be explicitly stated; the codomain should be as well (although it's possible that the codomain is inferable from context).

Do We Have to Show That...?

A good rule of thumb is: if you have to ask if you have to show it, then you have to show it. A more detailed method to determine whether you have to prove an intermediate result is summarized in the following flow chart.



More Info

This document was influenced by this article, which wonderfully summarizes many stylistic guidelines not mentioned here.