Homework 5 for Math 140C - Spring 2018

(I) Denote by $|A|$ the cardinality of the set A.

(a) Let $\mathcal{M} := \{A \subset \mathbb{Z} : |A| < \infty\}$. Determine whether $\mathcal{M}$ is a ring, a σ-ring, or neither. Rigorously justify your answer.

(b) Now, let $\mathcal{M} := \{A \subset \mathbb{Z} : |A| < \infty \text{ or } |\mathbb{Z} \setminus A| < \infty\}$. Determine whether $\mathcal{M}$ is a ring, a σ-ring, or neither. Rigorously justify your answer.

(II) Show that a $\sigma-$ring of subsets of a set X is either finite, or has uncountably many elements.

(III) Given any collection C of subsets of X, show that there is a smallest ring R containing C.

(IV) Let $\mathcal{E}$ be the family of elementary subsets of $\mathbb{R}$. Denote by $m : \mathcal{E} \to [0, \infty)$ the Lebesgue set function, and by $m^*(A)$ the outer measure of a set $A \subset \mathbb{R}$. For a scalar $t \in \mathbb{R}$ let the set $A + t := \{x + t | t \in A\}$.

(a) Show that if $A \in \mathcal{E}$, then $m(A + t) = m(A)$ for all $t \in \mathbb{R}$.

(b) Show that for an arbitrary $A \subset \mathbb{R}$, $m^*(A + t) = m^*(A)$ for all $t \in \mathbb{R}$.