Math 140C Midterm #2 Solutions

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Problem 1

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = ((x + z)^2 - y, -x^2 + y - z)$.

a. Prove that $f \in C^1$.

Solution The derivative of f is given by

$$Df = \begin{bmatrix} 2(x+z) & -1 & 2(x+z) \\ -2x & 1 & -1 \end{bmatrix}.$$

All of the partials are polynomials, so they are continuous. We have a theorem which states that a function is C^1 if and only if its partials exist and are continuous. This applies here, so $f \in C^1$.

b. Prove that there is an open set $W \subset \mathbb{R}$ with $1 \in W$, and a C^1 function $g : W \to \mathbb{R}^2$ such that $g(1) = (1, 0)$ and $f(x, g(x)) = (0, 0)$ for all $x \in W$.

Solution The Implicit Function Theorem states that, for an open $E \subset \mathbb{R}^{m+n}$ and $f : E \to \mathbb{R}^m$, if $f \in C^1$, then if $f(\mathbf{x}_0, \mathbf{y}_0) = \vec{0}$ for some $(\mathbf{x}_0, \mathbf{y}_0) \in E$ and if $A_\mathbf{x}$ is invertible (where $A_\mathbf{x} \in \mathbb{R}^{m\times m}$ is the restriction of $Df(\mathbf{x}_0, \mathbf{y}_0) = [A_\mathbf{x}\ A_\mathbf{y}]$ to its first m variables), then there exists an open set $W \subset \mathbb{R}^m$ and a C^1 function $g : W \to \mathbb{R}^n$ such that $\mathbf{y}_0 \in W, f(g(\mathbf{y}), \mathbf{y}) = \vec{0}$ for all $\mathbf{y} \in W$, and $g(\mathbf{y}_0) = \mathbf{x}_0$. Therefore, since we have already shown that $f \in C^1$, it suffices to show that $f(1, 1, 0) = \vec{0}$ and that A_{yz} is invertible at $(1, 1, 0)$. Indeed,

$$f(1, 1, 0) = ((1 + 0)^2 - 1, -1^2 + 1 - 0) = (0, 0)$$

and

$$A_{yz} = \begin{bmatrix} -1 & 2(1+0) \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix},$$

which has an inverse of

$$A_{yz}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Therefore, the Implicit Function Theorem gives all the desired conclusions.

c. Find the partial derivatives $\frac{\partial g_1}{\partial x}|_{x=1}$ and $\frac{\partial g_2}{\partial x}|_{x=1}$.

Solution The Implicit Function Theorem further gives that the resultant g has derivative given by

$$Dg = -A_\mathbf{x}^{-1} A_\mathbf{y} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

so the partials are $\frac{\partial g_1}{\partial x}|_{x=1} = 2$ and $\frac{\partial g_2}{\partial x}|_{x=1} = 0$.

Problem 2 Denote by $|A|$ the cardinality of the set A. We will consider all sets to be subsets of $\mathbb{Z}$, so that $A^c := \mathbb{Z} \setminus A$.

a. Let $\mathcal{M} := \{A \subset \mathbb{Z} : |A| < \infty\}$. Determine whether $\mathcal{M}$ is a ring, a σ-ring, or neither.

Solution $\mathcal{M}$ is not a σ-ring, as $\cup_{n=1}^{\infty}\{n\} \notin \mathcal{M}$, but $\{n\} \in \mathcal{M}$ for each n. However, $\mathcal{M}$ is a ring, as for $A, B \in \mathcal{M}$, we have $|A \cup B| \leq |A| + |B| < \infty$ and $|A \setminus B| = |A \cap B^c| \leq |A| < \infty$.

b. Let $\mathcal{M} = \{A \subset \mathbb{Z} : |A| < \infty \text{ or } |\mathbb{Z} \setminus A| < \infty\}$. Determine whether $\mathcal{M}$ is a ring, σ-ring, or neither.

Solution With two types of sets and two binary operations ($\cup$ and $\setminus$), we have 8 inclusions to prove to show that $\mathcal{M}$ is a ring:

	A	B	Op.	Justification
1.	$\lvert A\rvert < \infty$	$\lvert B\rvert < \infty$	$\cup$	Part (a)
2.	$\lvert A\rvert < \infty$	$\lvert B\rvert < \infty$	$\setminus$	Part (a)
3.	$\lvert A\rvert < \infty$	$\lvert B^c\rvert < \infty$	$\cup$	$\lvert(A \cup B)^c\rvert = \lvert A^c \cap B^c\rvert \leq \lvert B^c\rvert < \infty$
4.	$\lvert A\rvert < \infty$	$\lvert B^c\rvert < \infty$	$\setminus$	$\lvert A \setminus B\rvert = \lvert A \cap B^c\rvert \leq \lvert A\rvert < \infty$
5.	$\lvert A^c\rvert < \infty$	$\lvert B\rvert < \infty$	$\cup$	Same as 3.
6.	$\lvert A^c\rvert < \infty$	$\lvert B\rvert < \infty$	$\setminus$	$\lvert(A \setminus B)^c\rvert = \lvert(A \cap B^c)^c\rvert = \lvert A^c \cup B\rvert \leq \lvert A^c\rvert + \lvert B\rvert < \infty$
7.	$\lvert A^c\rvert < \infty$	$\lvert B^c\rvert < \infty$	$\cup$	$\lvert(A \cup B)^c\rvert = \lvert A^c \cap B^c\rvert \leq \lvert A^c\rvert < \infty$
8.	$\lvert A^c\rvert < \infty$	$\lvert B^c\rvert < \infty$	$\setminus$	$\lvert A \setminus B\rvert = \lvert A \cap B^c\rvert \leq \lvert B^c\rvert < \infty$

Therefore, $\mathcal{M}$ is a ring. However, the example of $A := \cup_{n=1}^{\infty}\{n\}$ is also a countable union of elements of $\mathcal{M}$; however, $|A| = \infty$ *and* $|A^c| = |\{0, -1, -2, \ldots\}| = \infty$, so $A \notin \mathcal{M}$.

Problem 3 Let $K_{\mathbb{Q}}$ be the characteristic function on the rationals; show that $K_{\mathbb{Q}}$ is Lebesgue measurable on $[0, 1]$. That is, show that $\forall a \in \mathbb{R}, \{x \in [0, 1] : f(x) > a\} \in \mathcal{M}(m)$, where m is the Lebesgue measure.

Solution Let all the terms be as defined, and let $E_a := \{x \in [0, 1] : f(x) > a\}$. We consider that when $a < 0, E_a = [0, 1]$, which is a closed set and hence Lebesgue measurable. For $0 \leq a < 1, E_a = \mathbb{Q} \cap [0, 1]$, which is countable and hence measurable (in particular, if $\{q_k\}$ enumerates $\mathbb{Q} \cap [0, 1]$, then $\mathbb{Q} \cap [0, 1] = \cup_{k=1}^{\infty}\{q_k\}$, is a countable union of closed sets). For $a \geq 1, E_a = \emptyset$ which is also trivially measurable.

Problem 4 Let $(X, \mathcal{M}, \mu)$ be a measure space with $X \in \mathcal{M}$ and suppose that $E_n \in \mathcal{M}, n = 1, 2, \ldots$ are such that $\mu(E_n^c) = 0$.

a. Let $E = \cap_{n=1}^{\infty} E_n$ and show that $\mu(E^c) = 0$.

Solution Using the non-negativity and subadditivity of measures, we have

$$0 \leq \mu(E^c) = \mu(\cup_{n=1}^{\infty} E_n^c) \leq \sum_{n=1}^{\infty} \mu(E_n^c) = 0,$$

so $\mu(E^c)$ and we are done.

b. Let $F = \{A \in \mathcal{M} : \mu(A) = 0 \text{ or } \mu(A^c) = 0\}$. Show that F is a σ-ring.

Solution Since $X \in \mathcal{M}$, it suffices to show that $\mathcal{F}$ is closed under countable intersections and complements (as $\cup A_n = (\cap A_n^c)^c$ and $A \setminus B = A \cap B^c$). Complements are immediate: for $A \in \mathcal{F}$, if $\mu(A) = 0$, then $A^c \in \mathcal{F}$ since $\mu((A^c)^c) = \mu(A) = 0$, and if $\mu(A^c) = 0$ then $A^c \in \mathcal{F}$ automatically.

For countable intersections, if $\{A_n\}_{n=1}^{\infty}$ includes at least one set A_N with $\mu(A_N) = 0$, then $0 \leq \mu(\cap_{n=1}^{\infty} A_n) \leq \mu(A_N) = 0$, so $\mu(\cap_{n=1}^{\infty} A_n) = 0$ and $\cap_{n=1}^{\infty} A_n \in \mathcal{F}$. The only other case is that $\mu(A_n^c) = 0$ for every n; by part (a), $\mu((\cap_{n=1}^{\infty} A_n)^c) = 0$, so $\cap_{n=1}^{\infty} A_n \in \mathcal{F}$.

This completes the proof that $\mathcal{F}$ is a σ ring.