Math 140C Midterm #1 Solutions

Solutions prepared by Brian Preskitt

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Problem 1 For $t > 0$, consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$(x, y) \to \left(\frac{\cos x + \sin y}{t}, \frac{\sin x - \cos y}{t} \right).$$

1. Find an upper bound for $||f'(x, y)||$. **Hint:** For a linear operator $A \in L(\mathbb{R}^2, \mathbb{R}^2)$ whose 2×2 matrix representation is

$$[A] := \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

it holds that $||A|| \leq \sqrt{a^2 + b^2 + c^2 + d^2}$.

Solution Well, the derivative of f is given by

$$Df = \begin{bmatrix} -\frac{\sin x}{t} & \frac{\cos y}{t} \\ \frac{\cos x}{t} & \frac{\sin y}{t} \end{bmatrix},$$

so by the hint we have

$$\begin{aligned} ||Df|| &\leq \sqrt{\frac{\sin^2 x + \cos^2 y + \cos^2 x + \sin^2 y}{t^2}} \\ &= \frac{\sqrt{2}}{t}, \end{aligned}$$

so our upper bound is that $||Df|| \leq \sqrt{2}/t$.

To prove the hint, we will consider the general case of $A \in L(\mathbb{R}^n)$; in this case, let $a_i \in \mathbb{R}^n$ be the columns of A (when represented with respect to the standard basis). Then for $x \in \mathbb{R}^n$, we have

$$|Ax| = |\sum_{i=1}^{n} x_i a_i| \leq \sum_{i=1}^{n} |x_i||a_i|.$$

We then apply the Cauchy-Schwarz inequality to the vectors u, v where $u_i = |x_i|$ and $v_i = |a_i|$ to get that

$$|Ax| \leq \sum_{i=1}^{n} |x_i||a_i| = u \cdot v \leq |u|\,|v| = |x| \sqrt{\sum_{i=1}^{n} |a_i|^2} = |x| \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}.$$

This coincides with the hint when $n = 2$.

Remark This last quantity $\sqrt{\sum_{i,j=1}^{n} a_{ij}^2} =: ||A||_F$ is called the *Frobenius norm* of the matrix A, named for the German mathematician Georg Frobenius (1849-1917). It is induced by the inner product

$$\langle A, B \rangle := \sum_{i,j=1}^{n} a_{ij} b_{ij} = \vec{A} \cdot \vec{B} = \mathrm{Tr}(A^T B)$$

and has the property that, for symmetric matrices,

$$||A||_F^2 = \sum_{i=1}^{n} \lambda_i^2,$$

where the λ_i denote the eigenvalues of A. Symmetric (or really Hermitian) matrices are wonderful – this is a great article on the subject of Hermitian matrices by Terry Tao.

2. For what values of t can you guarantee that f has a fixed point?

   Solution We want to use the contraction principle, which states that for any complete metric space X and any continuous mapping $\phi : X \to X$ satisfying $d(\phi(x), \phi(y)) \leq cd(x, y)$ for some constant $c < 1$, we have that there is a unique x such that $\phi(x) = x$. $\mathbb{R}^2$ is a complete metric space, so it suffices to find which values of t will guarantee that f is a contraction of $\mathbb{R}^2$. For this, we consider the consequence of the mean value theorem which gives that for any C^1 function on a convex open set with a derivative bounded by $||Df|| \leq M$, we have that $|f(x) - f(y)| \leq M|x - y|$. Certainly $\mathbb{R}^2$ is convex and open, so we can get f to be a contraction whenever $\sqrt{2}/t < 1$, that is, whenever $t > \sqrt{2}$.

Problem 2 Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator with $||A|| \leq \alpha < 1$.

1. Show that $I - A$ is invertible.

   Solution $I - A$ is non-invertible iff there is some non-zero x such that $(I - A)x = 0$, iff $Ax = x$. Of course this cannot happen, as $|Ax| \leq \alpha|x| < |x|$, so $Ax \neq x$ for all $x \neq 0$, so $I - A$ is invertible.

2. Find an upper bound on $||(I - A)^{-1}||$.

   Solution We consider the definition of the operator norm:

$$\begin{aligned} ||(I - A)^{-1}|| &= \sup_{x \neq 0} \frac{|(I - A)^{-1}x|}{|x|} \\ &= \sup_{y \neq 0} \frac{|(I - A)^{-1}(I - A)y|}{|(I - A)y|} \\ &= \sup_{y \neq 0} \frac{|y|}{|(I - A)y|} \end{aligned}$$

   We now consider that, for any y, $|(I - A)y| \geq |y| - |Ay| \geq |y| - \alpha|y| = (1 - \alpha)|y|$, which allows us to write

$$\begin{aligned} ||(I - A)^{-1}|| &= \sup_{y \neq 0} \frac{|y|}{|(I - A)y|} \\ &\leq \sup_{y \neq 0} \frac{|y|}{(1 - \alpha)|y|} \\ &= \frac{1}{1 - \alpha}, \end{aligned}$$

   so our upper bound is $||(I - A)^{-1}|| \leq \dfrac{1}{1 - \alpha}$. (Note that this upper bound is always achievable, for example if $A = \alpha I$).

3. For what values of β are all matrices $I - B$ satisfying $||B - A|| \leq \beta$ invertible?

   Solution We will attempt to cite the theorem which says that, for operators P and Q, if Q is invertible and we have $||P - Q||\,||Q^{-1}|| < 1$, then P is invertible. Considering that $||B - A|| = ||(I - B) - (I - A)||$, we quote that result with $P = (I - B)$ and $Q = (I - A)$ to have that $I - B$ is invertible whenever $||B - A||\,||(I - A)^{-1}|| < 1$. From our result in (b), this is satisfied when $||B - A|| < 1 - \alpha$, so any β satisfying $\beta < 1 - \alpha$ will work.

Problem 3 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$(x, y) \to ((x + y)^2, x^4 + y^4).$$

1. Show that f is continuously differentiable.

   Solution The derivative is
   $$Df = \begin{bmatrix} 2(x+y) & 2(x+y) \\ 4x^3 & 4y^3 \end{bmatrix},$$
   all of whose entries are clearly existant and continuous, so $f \in C^1$.

2. Show that there is an open set containing $(x, y) = (1, 0)$ on which f is invertible.

   Solution We quote the Inverse Function Theorem, which states that if $g \in C^1(E \subset \mathbb{R}^n, \mathbb{R}^n)$ for some open set E, then if $x \in E$ has that $Dg(x)$ is invertible, then there is an open set $x \in U \subset E$ on which g is invertible. In our case, f is C^1, its domain $\mathbb{R}^2$ is open, and
   $$Df(1, 0) = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$$
   is invertible (consider that its determinant is $-8 \neq 0$), so the IFT guarantees the existence of an open set containing $(1, 0)$ on which f is invertible.

3. Show that there exists *no* open set containing $(x, y) = (1, 1)$ on which f is invertible.

   Solution Consider that $f(x, y) = f(y, x)$ for every $(x, y) \in \mathbb{R}^2$. In particular, if $(1, 1) \in U$ is an open set then there is some $\epsilon > 0$ such that $B_\epsilon((1, 1)) \subset U$, so $(1, 1 + \epsilon/2)$ and $(1 + \epsilon/2, 1)$ are in U while $f(1, 1 + \epsilon/2) = f(1 + \epsilon/2, 1)$, which shows that f is not injective on U and therefore not invertible on U.

4. Why does this not contradict the inverse function theorem?

   Solution This does not contradict the IFT because the hypotheses of IFT are not satisfied – in particular,
   $$Df(1, 1) = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix},$$
   which is not invertible as $(1, -1)$ is in its null space.