Surface Area Notes

Math 31CH, Spring 2016

In this week's section (of Monday, 5/2/2016), the question was raised:

What is the geometric interpretation of forms?

The response I gave was something to the effect of

Forms give us a language with which we may speak of many geometric phenomena in Euclidean space,

which is certainly a vague answer! I tried to give an example of this, namely by sharing how one can use forms to calculate the surface area of a manifold; namely, I shared a claim I found that the following form measures surface area:

Claim Let $M \subset \mathbb{R}^n$ be an orientable, $(n-1)$-dimensional smooth manifold in $\mathbb{R}^n$, and let $\eta : M \to \mathbb{S}^{n-1}$ be a continuous mapping such that $\eta(\mathbf{x}) \perp T_{\mathbf{x}}M$ (that is, $\eta(\mathbf{x})$ is a unit normal vector to M at $\mathbf{x}$). Then if ω is the $(n-1)$-form field on M defined by

$$\omega_{\mathbf{x}}(v_1, \ldots, v_{n-1}) = \det\left[\eta(\mathbf{x})\ v_1\ \cdots\ v_{n-1}\right],$$

then $\int_M \omega$ gives the surface area ($n-1$-dimensional volume) of M.

Remark Consider that this form is defined up to a factor of ± 1, as there are two continuous normal vector fields η; namely, if η is a continuous normal vector field, then so is $-\eta$! To resolve this ambiguity, we specify an orientation Ω of M and choose η such that $\Omega = \operatorname{sgn} \omega$, as on p. 578 of the text.

I tried to demonstrate this claim with an example: namely, we explicitly calculated η and ω for the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, but this is hardly a proof that this form does what it promises, and certainly gave little intuition as to "why." At any rate, the proof that ω is indeed the $(n-1)$-dimensional volume form in $\mathbb{R}^n$ is pretty quick! We will show that ω coincides with **Definition 5.3.1** from the textbook (p. 533), the volume of a parametrized manifold. Considering from **Theorem 5.2.8** (p. 530) that every manifold can be parametrized, we will conclude that ω correctly expresses the $(n-1)$-dimensional volume of a manifold in terms of the integral of a form!

Proposition Given an $(n-1)$-dimensional, orientable, smooth manifold $M \subset \mathbb{R}^n$, an orientation-preserving (p. 584) parametrization $\gamma : U \to \mathbb{R}^n$ of M, the mapping η defined above, and the form ω defined above, then

$$\int_M \omega = \int_U \sqrt{\det\left([D\gamma(\mathbf{u})]^T[D\gamma(\mathbf{u})]\right)}\ |d\mathbf{u}^{n-1}|.$$

Proof According to **Definition 6.4.12** (p. 592), we have

$$\begin{aligned}\textstyle\int_M \omega &= \textstyle\int_U \omega\left(P_{\gamma(\mathbf{u})}\left(D_1\gamma(\mathbf{u}),\ldots,D_{n-1}\gamma(\mathbf{u})\right)\right)\ |d^{n-1}\mathbf{u}| \\ &= \textstyle\int_U \det\left[\eta(\gamma(\mathbf{u}))\ D_1\gamma\ \cdots\ D_{n-1}\gamma\right]\ |d^{n-1}\mathbf{u}|.\end{aligned}$$

We now consider that $D_j\gamma(\mathbf{u}) \in T_{\gamma(\mathbf{u})}M$ for every $\mathbf{u} \in U, j = 1,\ldots,n-1$. Indeed, this is equivalent to the definition of $T_{\gamma(\mathbf{u})}M$. In particular, we have $D_j\gamma(\mathbf{u})^T\eta(\gamma(\mathbf{u})) = 0$, so, setting $A(\mathbf{u}) = [\eta(\gamma(\mathbf{u}))\ D_1\gamma\ \cdots\ D_{n-1}\gamma]$ and recalling that we oriented ω and γ such that $\det A(\mathbf{u}) > 0$, we have

$$\begin{aligned}\det A(\mathbf{u}) &= \sqrt{\det(A(\mathbf{u})^T)\det(A(\mathbf{u}))} \\ &= \sqrt{\det\left(\begin{bmatrix}\eta^T \\ D_1^T \\ \vdots \\ D_{n-1}^T\end{bmatrix}\begin{bmatrix}\eta & D_1 & \cdots & D_{n-1}\end{bmatrix}\right)} \\ &= \sqrt{\det\left(\begin{bmatrix}1 & 0 \\ 0 & D\gamma^T D\gamma\end{bmatrix}\right)} \\ &= \sqrt{\det\left(D\gamma^T D\gamma\right)}.\end{aligned}$$

Combining with the previous calculation gives

$$\int_M \omega = \int_U \sqrt{\det\left(D\gamma(\mathbf{u})^T D\gamma(u)\right)}\ |d^{n-1}\mathbf{u}|$$

as promised! □

Obviously, the key part of this last proof is that we can somehow get $\sqrt{\det(D\gamma^T D\gamma)}$, the volume element from the parametrization, from the form ω. This proof technique easily lends itself to devising a k-dimensional volume form for any k, not just $k = n-1$! In order to state this form concisely, we introduce the following term:

Definition Given a k-form field $\phi : M \to A_c^k(\mathbb{R}^n)$ and a vector field $\chi : M \to TM$, the **contraction** of ϕ with χ is the $(k-1)$-form field $\iota_\chi\phi$ defined by

$$\iota_\chi\phi(\mathbf{x})(v_1,\ldots,v_{k-1}) = \phi(\chi(\mathbf{x}), v_1,\ldots,v_{k-1}).$$

In other words, the contraction of a form ϕ with a vector field χ is just ϕ with the first argument always chosen to be χ!

With this notation, it is clear that if $\nu \in A^n(\mathbb{R}^n)$ is the basic volume form (namely $\nu(v_1,\ldots,v_n) = \det[v_1\ \cdots\ v_n]$), then

$$\omega = \iota_\eta\nu.$$

Following this example, I will leave it as an exercise for those interested that if $(\eta_1, \ldots, \eta_{n-k}) : M \to TM^\perp$ is an appropriately oriented, continuous collection of maps such that $\{\eta_i(\mathbf{x})\}_{i=1}^{n-k}$ forms an orthonormal basis for $T_{\mathbf{x}}M^\perp$ (the orthogonal complement of the tangent space) at each $\mathbf{x} \in M$, then the k-dimensional volume of M may be found by taking $\omega^{(k)} = \iota_{\eta_{n-k}} \cdots \iota_{\eta_1} \nu$, i.e.

$$\omega^{(k)}(\mathbf{x})(v_1, \ldots, v_k) = \det\left[\eta_1 \ \cdots \ \eta_{n-k} \ v_1 \ \cdots \ v_k\right],$$

and integrating

$$\mathrm{vol}_k M = \int_M \omega^{(k)}.$$

The proof comes easily by following the technique used above.

Enjoy!