Bounded Orthonormal Systems Continued:

Theorem: $m \geq C K^2 \delta^{-2} s \ln^4(N)$
$\Rightarrow$ w.p. $\geq N^{-\ln^3 N}$: $\delta_s \leq \delta$

We have seen that the "standard" strategy of bounding the RIP constant for one submatrix A_S $S \subset [N]$, $|S| \leq s$ and then using a union bound over all $\binom{N}{s}$ submatrices fails.

Instead, we will use Dudley's inequality (more later)

$$\mathbb{E} \sup_{t \in T} X_t \leq 4\sqrt{2} \int_0^{\Delta(T)/2} \sqrt{\ln N(T, d, u)}\, du$$
$$\Delta(T) = \sup_{t \in T} \sqrt{\mathbb{E}|X_t|^2}$$
$$d(u,v) = \left(\mathbb{E}|X_u - X_v|^2\right)^{1/2}$$

Proof of Theorem:

note that $\delta_s := \sup_{\substack{S \subset [N] \\ |S| \leq s}} \| A_S^* A_S - I \|_{2 \to 2}$

$A = \frac{1}{\sqrt{m}} R U \to$ BOS, R: $m \times N$ random test

and define the seminorm

$$|||B|||_s := \sup_{z \in \Sigma_s^N \cap B_2^N} |\langle Bz, z \rangle|$$

so that $\delta_s = |||A^*A - I|||_s$

Define X_ℓ^*, the ℓth row of $\sqrt{m}A$,
so that $\mathbb{E} X_\ell^* X_\ell = I$ and $\sum_{\ell=1}^m X_\ell X_\ell^* = A^*A$
(BOS)

Thus $\delta_s = \| \frac{1}{m} \sum_{\ell=1}^m X_\ell X_\ell^* - I \|_s = \frac{1}{m} \| \sum X_\ell X_\ell^* - \mathbb{E} X_\ell X_\ell^* \|_s$

Step 1 $\mathbb{E}\delta_s$

Now, let us consider

$$E := \mathbb{E}\delta_s = \mathbb{E} \| \sum_{\ell=1}^m \left(X_\ell X_\ell^* - \mathbb{E} X_\ell X_\ell^* \right) \|_s / m$$

$$= 2\mathbb{E} \| \sum_{\ell=1}^m \epsilon_\ell X_\ell X_\ell^* \|_s / m$$

HW (Symmetrization)

± 1 iid Bernoulli

Lemma: Let $x_1 \dots x_m \in \mathbb{C}^N$, $\|x_\ell\|_\infty \le K$
Then for $s \le m$

$$\mathbb{E}_\epsilon \| \sum_{\ell=1}^m \epsilon_\ell x_\ell x_\ell^* \|_s \le C_1 K \sqrt{s} \ln(4s) \sqrt{\ln(8N)\ln(9m)} \sqrt{\| \sum_{\ell=1}^m x_\ell x_\ell^* \|_s}$$

Lemma (a)

Set $G(K,\delta,m,N) = K\sqrt{\delta}\ln(4/\delta)\sqrt{\ln(8N)\ln(9m)}$

Now,

$$E = \frac{2}{m}\mathbb{E}\left\|\sum_{\ell=1}^{m}\epsilon_\ell X_\ell X_\ell^*\right\|_s \leq \frac{2}{m}C_1 G\,\mathbb{E}_X\sqrt{\left\|\sum_{\ell=1}^{m}X_\ell X_\ell^*\right\|_s}$$

(over ϵ; lemma (a))

$$\leq \frac{2}{\sqrt{m}}C_1 G\,\mathbb{E}\sqrt{\left\|\frac{\sum_{\ell=1}^{m}X_\ell X_\ell^*}{m}\right\|_s}$$

$$\leq \frac{2}{\sqrt{m}}C_1 G\,\mathbb{E}\sqrt{\left\|\frac{\sum(X_\ell X_\ell^* - I + I)}{m}\right\|_s}$$

$$\leq \frac{2}{\sqrt{m}}C_1 G\,\mathbb{E}\sqrt{\underbrace{\frac{\sum(X_\ell X_\ell^* - I)}{m}}_{=E} + 1}$$

(Δ-ineq.)

$$\leq \frac{2}{\sqrt{m}}C_1 G\sqrt{E+1}$$

($\mathbb{E}\sqrt{a+b} \leq \sqrt{\mathbb{E}(a+b)}$)

So $E \leq D\sqrt{E+1}$ for $D = \frac{2}{\sqrt{m}}C_1 G$

$\Rightarrow$ $E \leq D + D^2$, where

$$D = 2C_1 K\sqrt{\frac{2s}{m}}\ln(4/\delta)\sqrt{\ln(9m)}\sqrt{\ln(8N)}$$

Step 2: $\mathbb{P}(|\xi - \mathbb{E}\xi| > ?) < ?$

We will use a (nother) Bernstein inequality

Let $\mathcal{F}$ be a countable set of functions
$F : \mathbb{C}^n \to \mathbb{R}$
and let $Y_1 \dots Y_m$ be ind. r.vectors in $\mathbb{C}^n$
with $\mathbb{E}(Y_\ell) = 0$, $F(Y_\ell) \le K$ a.s.

Let $Z = \sup_{F \in \mathcal{F}} \sum_{\ell=1}^{m} F(Y_\ell)$

Then
$$\mathbb{P}(Z \ge \mathbb{E}Z + t) \le \exp\left(\frac{-t^2/2}{\sigma^2 + 2K\mathbb{E}Z + \frac{tK}{3}} \right)$$
where $\sigma^2 = \sum_{\ell=1}^{m} \sigma_\ell^2$ and $\sigma_\ell^2 \ge \mathbb{E}[F(Y_\ell)^2]$

Bernstein inequality for empirical processes.

Now, Recall $\delta_S = \frac{1}{m} \left\| \sum_{\ell=1}^{m} (X_\ell X_\ell^* - I) \right\|_S$

Define $Q_S = \{ (z, w) \in \mathbb{C}^N \times \mathbb{C}^N, \|z\|_2 = \|w\|_2 = 1$
$\operatorname{supp}(z), \operatorname{supp}(w) \subset S \}$

and $Q_s = \bigcup_{\substack{S \subset [N] \\ |S| \le s}} Q_S$

so $\delta_s = \frac{1}{m} \sup_{\substack{S \subset [N] \\ |S| \le s}} \left\| \sum_{\ell=1}^{m} \left[(X_\ell)_S (X_\ell^*)_S - I \right] \right\|_{2 \to 2}$

$$= \frac{1}{m} \sup_{(z,w) \in Q_s} \mathrm{Re} \left\langle \sum \left(X_\ell^* X_\ell - I \right) z, w \right\rangle$$

$$= \frac{1}{m} \sup_{\substack{(z,w) \\ \in Q_s}} \sum_{\ell=1}^{m} \mathrm{Re} \left\langle \left(X_\ell X_\ell^* - I \right) z, w \right\rangle$$

$$= \frac{1}{m} \sup_{\substack{(z,w) \\ \in Q_s^*}} \sum_{\ell=1}^{m} \underbrace{\mathrm{Re} \left\langle \left(X_\ell X_\ell^* - I \right) z, w \right\rangle}_{=: f_{z,w}(X_\ell)}$$

dense countable subset of Q_s (need this for Bernstein)

Remains to check boundedness of $f_{z,w}$ before applying Bernstein.

$$|f_{z,w}(X)| = \mathrm{Re} \left\langle (XX^* - I) z, w \right\rangle$$

$$\leq \|X_S X_S^* - I\|_{2\to 2} \underbrace{\|z\|_2 \|w\|_2}_{1}$$

HW ↓
$$\leq \|X_S X_S^* - I\|_{1\to 1}$$

HW ↓
$$= \max_{j\in S} \sum_{k\in S} |\varphi_j(t)\overline{\varphi_k(t)} - \delta_{jk}|$$

$$\leq K^2 s$$

Still need $\sigma_i^2 \geq \mathbb{E}|F_{z,w}(X_\ell)|^2$

but

$$\mathbb{E}|F_{z,w}(X_\ell)|^2 \leq \mathbb{E}|\langle (XX^* - I)z, w\rangle|^2$$

(z supported on S)

$$\leq \underbrace{\|w\|_2^2}_{1} \mathbb{E}\|(X_S X_S^* - I)z\|_2^2$$

$$= \mathbb{E}\|X_S\|_2^2 |\langle X_S, z\rangle|^2 - 2\mathbb{E}|\langle X_S, z\rangle|^2 + 1$$

$\mathbb{E}\|X_S\|_2^2 = \sum_{j\in S} |\varphi_j(t)|^2 \leq K^2 s$

$\mathbb{E}|\langle X_S, z\rangle|^2 = \mathbb{E}\left|\sum_{j\in S} \varphi_j(t) z_j\right|^2 = \|z\|_2^2 = 1$

(cross terms cancel + BOS)

$$\leq K^2 s - 1 \leq K^2 s$$

Finally by Bernstein:

$$\mathbb{P}\left(\delta_s \geq \eta_1 + \eta_1^2 + \eta_2\right) \leq \mathbb{P}\left(\delta_s \geq \mathbb{E}\delta_s + \eta_2\right)$$

$\eta_1 \geq 0$

$$= \mathbb{P}\left(\left\|\left|\Sigma(X_\ell X_\ell^* - I)\right\|\right|_s \geq \mathbb{E}\left\|\left|\Sigma(X_\ell X_\ell^* - I)\right\|\right|_s + \eta_2 m\right)$$

$$\leq \exp\left(\frac{-(\eta_2 m)^2}{2ms\|K\|^2 + 4(\eta_1+\eta_1^2)ms\|K\|^2 + 2\eta_2 ms\|K\|^2/3}\right)$$

$$\leq \exp\left(-\underbrace{c(\eta_1)}_{\leq 3/32}\frac{m\eta_2^2}{\|K\|^2 s}\right)$$

$$\leq \varepsilon \quad \text{when} \quad \underbrace{m \geq C\eta_2^{-2}\|K\|^2 s \ln(1/\varepsilon)}_{\text{cond 1}}$$

But we also need

$$\eta_1 \geq D = \frac{2}{\sqrt{m}} C_1 \|K\| \sqrt{s}\ln(4s)\sqrt{\ln(8N)\ln(9m)}$$

i.e

$$\underbrace{m \geq C\eta_1^{-2}\|K\|^2 s \ln^2(4s)\ln(8N)}_{\text{cond 2}}$$

∎

Remark: Remains to Prove Lemma (a). (We won't prove Bernstein)

We will next prove the Lemma, which relies on Dudley's inequality and then we will discuss chaining techniques (and prove Dudley's inequality.

Lemma: Let $x_1 \dots x_m \in \mathbb{C}^N$, $\|x_\ell\|_\infty \le K$

Then $\forall s \le m$

$$\mathbb{E}_\epsilon \left\| \sum_{\ell=1}^m \epsilon_\ell x_\ell x_\ell^* \right\|_s \le C_1 K \sqrt{s} \ln(4s) \sqrt{\ln(8N)\ln(9m)} \sqrt{\left\| \sum_{\ell=1}^m x_\ell x_\ell^* \right\|_s}$$

→ Lemma (a)

Proof: Let $E := \mathbb{E} \underbrace{\left\| \sum_{\ell=1}^m \epsilon_\ell x_\ell x_\ell^* \right\|_s}_{\|B\|_s = \sup_{u \in \Sigma_s^N \cap B_2^N} |\langle Bu, u\rangle|} = \mathbb{E} \sup_{u \in \Sigma_s^N \cap B_2^N} \left| \sum_{\ell=1}^m \epsilon_\ell K \langle x_\ell, u\rangle^2 \right|$

The idea is to use dudley's inequality:

$$\mathbb{E} \sup_{t \in T} X_t \le 4\sqrt{2} \int_0^{\Delta(T)/2} \sqrt{\ln N(T, d, u)}\, du$$

where: $\begin{cases} \Delta(T) = \sup_{t \in T} \sqrt{\mathbb{E}|X_t|^2} & \text{is the radius of } T \\ d(u,v) = (\mathbb{E}|X_u - X_v|^2)^{1/2} & \text{is the pseudometric} \end{cases}$

associated with X_t.

So, we define the Rademacher Process

$$X_u = \sum_{\ell=1}^{m} \epsilon_\ell |\langle x_\ell, u\rangle|^2 \quad \text{with} \quad T = \Sigma_S^N \cap B_2^N$$

and its pseudometric

$$d(u,v) = \mathbb{E}\left(|X_u - X_v|^2\right)^{1/2} = \sqrt{\sum_{\ell=1}^{m} \underbrace{\epsilon_\ell^2}_{=1} \left(|\langle x_\ell, u\rangle|^2 - |\langle x_\ell, v\rangle|^2\right)^2}$$

$$= \sqrt{\sum_{\ell=1}^{m} \underbrace{\left(|\langle x_\ell, u\rangle| - |\langle x_\ell, v\rangle|\right)^2}_{a_\ell} \underbrace{\left(|\langle x_\ell, u\rangle| + |\langle x_\ell, v\rangle|\right)^2}_{b_\ell}}$$

$$\leq \sqrt{\|\vec{a}\|_\infty \, \|\vec{b}\|_1}$$

$$= \sup_\ell \left||\langle x_\ell, u\rangle| - |\langle x_\ell, v\rangle|\right| \cdot \sup_{u,v \in \Sigma_S^N \cap B_2^N} \sqrt{\sum_{\ell=1}^{m} |\langle x_\ell, u\rangle|^2 + |\langle x_\ell, v\rangle|^2}$$

$$\leq \max_\ell |\langle x_\ell, u - v\rangle| \cdot 2 \sup_{u \in \Sigma_S^N \cap B_2^N} \sqrt{\sum_{\ell=1}^{m} |\langle x_\ell, u\rangle|^2}$$

$$:= \|u - v\|_X \qquad R'' = \sqrt{\left\|\sum x_\ell x_\ell^*\right\|_S}$$

So

$$\left(\mathbb{E}\left|\frac{X_u}{2R} - \frac{X_v}{2R}\right|^2\right)^{1/2} \leq \|u - v\|_X$$

So, by Dudley's inequality:

$$E := \mathbb{E} \sup_{t \in T} X_t \leq C \int_0^{\Delta(T)/2} \sqrt{\ln N(T, d, u)}\, du$$

$$\leq 2CR \int_0^{\tilde{\Delta}(T)/2} \sqrt{\ln N(T, \|\cdot\|_X, t)}\, dt$$

Sec ① $N(T,d,u) \leq N(T, \|\cdot\|_X \cdot 2R, u) = N(T, \|\cdot\|_X, \underbrace{\tfrac{u}{2R}}_{t})$

$\Rightarrow du = 2R\,dt$

and ② $\tilde{\Delta}(\Sigma_s^N \cap B_2^N) := \sup_{u \in \Sigma_s^N \cap B_2^N} \|u\|_X \geq \Delta\left(\frac{T}{2R}\right)$

$$\Rightarrow E \leq 2CR \int_0^{\frac{\tilde{\Delta}}{2}(\Sigma_s^N \cap B_2^N)} \sqrt{\ln(2N(\Sigma_s^N \cap B_2^N, \|\cdot\|_X, t)}\, dt$$

(I)

(II)

(I)

But $\|u\|_X = \max_{\ell \in [m]} |\langle x_\ell, u\rangle| \leq \|u\|_1 \max \|x_\ell\|_\infty$

$\leq \sqrt{s}\,\|u\|_2\, K \leq K\sqrt{s}$

$\Rightarrow \Delta \leq K\sqrt{s}$

(II) Remains to estimate $N(\Sigma_s^N \cap B_2^N, \|\cdot\|_X, t)$

Two cases (II a) t is "small"

(II b) t is "large"

(IIa): $N(\Sigma_s^N \cap B_2^N, \|\cdot\|_X, t)$

$$\leq \sum_{\substack{T \subset [N] \\ |T| = s}} N(\Sigma_T^N \cap B_2^N, \underbrace{\|\cdot\|_X}_{\| \cdot \|_\Lambda}, t)$$

$$\leq \sum_{\substack{T \subset [N] \\ |T| = s}} N(\Sigma_T^N \cap B_2^N, \|\cdot\|_2 K\sqrt{s}, t)$$

$$\leq \binom{N}{s}\left(1 + \frac{2K\sqrt{s}}{t}\right)^{2s} \leq \left(\frac{eN}{s}\right)^s \left(1 + \frac{2K\sqrt{s}}{t}\right)^{2s}$$

(B_2 in $\mathbb{C}^n \equiv B_2$ in $\mathbb{R}^{2n}$)

$$\Rightarrow \sqrt{\ln\left(2N(\Sigma_s^N \cap B_2, \|\cdot\|_X, t\right)} \leq \sqrt{2s}\left(\sqrt{\ln \tfrac{2eN}{s}} + \sqrt{\ln\left(1 + \tfrac{2K\sqrt{s}}{t}\right)}\right)$$

(IIb): Let $\|z\|_1^* = \sum_{j=1}^{N} |\mathrm{Re}(z_j)| + |\mathrm{Im}(z_j)|$

$$\Rightarrow \Sigma_s^N \cap B_2^N \subset \sqrt{2s}\, B_{\|\cdot\|_1^*}^N$$

But

For $0 < t < 2K\sqrt{s}$

Lemma (b): $\sqrt{\ln\left(2N(\Sigma_s^N \cap B_2, \|\cdot\|_X, t\right)}$

$$\leq \frac{6K\sqrt{2s}}{t}\sqrt{\ln(9m)\ln(8N)}$$

Putting things together:

$$I := \int_0^{\Delta} \sqrt{\ln\left(2\mathcal{N}\left(\Sigma_S^N \cap B_2^N, \|\cdot\|_X, t\right)\right)}\, dt$$

$$\leq \sqrt{2s} \int_0^{\rho} \sqrt{\ln \tfrac{2eN}{s}} + \sqrt{\ln\left(1 + 2K\sqrt{s}/t\right)}\, dt$$

$$+ \int_{\rho}^{\underbrace{K\sqrt{s}/2}_{\geq \Delta/2}} \frac{6K\sqrt{2s}}{t} \sqrt{\ln(9m)\ln(8N)}\, dt$$

$$\int_0^{\alpha} \sqrt{\ln\left(1+\tfrac{1}{t}\right)} \leq \alpha \sqrt{\ln e\left(1+\tfrac{1}{\alpha}\right)}$$

$$\leq \sqrt{2s}\left(\rho\sqrt{\ln \tfrac{2eN}{s}} + \rho\sqrt{\ln\left(1 + 2K\sqrt{s}/\rho\right)}\right.$$

$$\left. + 6K\sqrt{\ln(9m)\ln(8N)}\ln\left(\frac{K\sqrt{s}/2}{\rho}\right)\right)$$

Pick $\rho = K/3$

$$\Rightarrow I \leq \sqrt{2s}\, K C \sqrt{\ln(9m)\ln(8N)}\ln 4s$$

So $E \leq 2CR \int_0^{\Delta(\Sigma_S^N \cap B_2^N)} \sqrt{\ln\left(2\mathcal{N}\left(\Sigma_S^N \cap B_2^N, \|\cdot\|_X, t\right)\right)}\, dt$

$$\leq 2C \sqrt{\left\|\sum_{\ell=1}^{m} x_\ell x_\ell^*\right\|_S}\, \sqrt{2s}\, K C \sqrt{\ln(9m)\ln(8N)}\ln 4s$$

$\blacksquare$

Remains to prove lemma (b) (in more general form)

Lemma (b): $\forall U \subset B^N_{\|\cdot\|_1^*}$, and $\|\cdot\|_X = \max_i |\langle x_i, \cdot\rangle|$

where $\|x_i\|_0 \leq k$:

$$\sqrt{\ln(2N(U, \|\cdot\|_X, t))} \leq \frac{6K\sqrt{\ln(en)\ln(8N)}}{t}$$

$\forall 0 < t < \sqrt{2}k$

Proof: Idea: "Empirical method of Maurey"

Note that $B^N_{\|\cdot\|_1^*} = \text{conv}\underbrace{\{\pm e_j, \pm ie_j, j \in [N]\}}_{V,\ |V| = 4N}$

$\Rightarrow \forall x \in U$, $\exists \lambda_j$ s.t. $x = \sum_{j=1}^{4N} \lambda_j v_j$ w/ $v_j \in V$, $\sum \lambda_j = 1$, $\lambda_j \geq 0$

Let Z be a random vector with

$$P(Z = v_j) = \lambda_j \quad \forall j \in [4N]$$

$\Rightarrow \mathbb{E}Z = \sum \lambda_j v_j = x$

Now, Let $Z_1, \ldots, Z_n \sim Z$

m ← TBD

and define $z = \sum_{j=1}^{m} z_j / m$

So each z_j is sparse and their $\mathbb{E} = x$ &
we are approximating $x \in U$ with z

$$\mathbb{E}\|z-x\|_X = \mathbb{E}\left\|\frac{1}{m}\sum_{j=1}^{m}(z_j - \mathbb{E}z_j)\right\|_X$$

$$\leq \frac{2}{m}\mathbb{E}\left\|\sum_{j=1}^{m}\epsilon_j z_j\right\|_X = \frac{2}{m}\mathbb{E}\max_{\ell\in[N]}\underbrace{\left|\sum_{j=1}^{m}\epsilon_j\langle z_j, x_\ell\rangle\right|}_{Y_\ell}$$

↑ symmetrization

Conditioning on $z_1, \ldots, z_m$ and noting that $z_j \in Z_I$

we see that $|\langle z_j, x_\ell\rangle| \leq K \cdot 1$ (Holder)

$$\Rightarrow \|(\langle z_j, x_\ell\rangle)_{j\in[N]}\|_2 \leq \sqrt{m}\,K$$

$$\Rightarrow \mathbb{P}_\epsilon(|Y_\ell| \geq \sqrt{m}\,K t) \leq 2e^{-t^2/2} \quad t \geq 0 \text{ (Hoeffding)}$$

$$\Rightarrow \mathbb{P}_\epsilon(\max_{\ell\in[N]}|Y_\ell| \geq \sqrt{m}\,K t) \leq 2Ne^{-t^2/2} \text{ (union bnd)}$$

$$\Rightarrow \mathbb{E}_\epsilon \max_{\ell\in[N]}|Y_\ell| \leq \frac{3}{2}\sqrt{m}\,K\sqrt{\ln(8N)}$$

↑ HW (see prop 7.14)

$$\Rightarrow \mathbb{E}\|z-x\|_X \leq \frac{2}{m}\mathbb{E}_\epsilon \max_{\ell\in[N]}\left|\sum\epsilon_\ell\langle z_\ell, x_\ell\rangle\right| \leq \frac{3}{m}K\sqrt{\ln 8N}$$

$M \quad z_\ell \in \ell \in [m] \mid k \quad \ldots \quad \sqrt{M}$

Since the $\mathbb{E}$ is bounded by

$\exists\, z = \sum_{k=1}^{M} z_k / M$, $z_k \in V$ with $\|z - x\|_X \leq$

$\Rightarrow \|z - x\|_X \leq t/2$ whenever $3K\sqrt{\frac{\ln 8m}{M}} \leq t/2$ — $(*)$

$\forall x \in U$, $\exists$ such a z. but there are only $|V|^M$
$= (4N)^M$ such z's in $B^N_{\|\ \|_1^*}$

The only remaining issue is that z_x may not be in U
• if it isn't then use $z' \in U$
where $\|z' - z\|_X \leq t/2$ whenever such
a z' exists, otherwise discard z.

now $\|x - z'\|_X \leq \|x - z\|_X + \|z - z'\|_X \leq t$

Pick $M = \left\lfloor \frac{36K^2}{t^2} \ln 9m \right\rfloor$ to satisfy $(*)$

and do the arithmetic with

$N(\mathcal{U}, \|\ \|_X, t) \leq (4N)^M$ to finish ∎