Review for midterm 1:

Let $g(u,v) = (uv^2, u+2v, v)$. Suppose $f: \mathbb{R}^3 \to \mathbb{R}^2$ with $f(0,0,0) = (1,2)$ and

$$Df(0,0,0) = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Let $h = g \circ f$. Find $Dh(0,0,0)$

Soln: $Dh|_x = \underbrace{Dg|_{f(x)}}_{3\times 2} \underbrace{Df|_x}_{2\times 3} = \begin{bmatrix} v^2 & 2uv \\ 2 & 2 \\ 0 & 1 \end{bmatrix}_{(1,2)} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 4 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 4 \\ 5 & 5 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

Evaluate $\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, dx\,dy\,dz$ where D: $1 \le x^2+y^2+z^2 \le 2$, $z \ge 0$

Spherical coord. $\Rightarrow \iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, dx\,dy\,dz = \iiint_{\substack{1\le\rho^2\le 2 \\ 0\le\phi\le\pi/2}} e^{\rho^3} \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$

$$= \int_0^{\pi/2}\int_0^{2\pi}\left(\int_1^{\sqrt{2}} e^{\rho^3}\rho^2 \sin\phi \, d\rho\right) d\theta\, d\phi = \int_0^{\pi/2}\int_0^{2\pi} \frac{1}{3} e^{\rho^3}\Big|_{\rho=1}^{\sqrt{2}} \sin\phi \, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \frac{2\pi}{3}\left(e^{\sqrt{8}} - e\right)\sin\phi \, d\phi = (-\cos\phi)\Big|_0^{\pi/2} \cdot \frac{2\pi}{3}\left(e^{\sqrt{8}} - e\right)$$

$$= (2\pi/3)\left(e^{\sqrt{8}} - e\right)$$

Let $G(x,y) = (xe^{x^2+y^2} + 2xy)\vec{i} + (ye^{x^2+y^2} + x^2)\vec{j}$

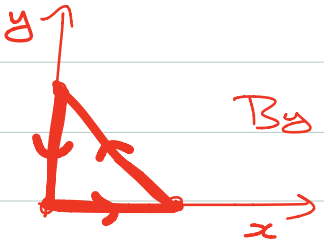
(a) Find an f so $G = \nabla f$.

Want f with $\frac{\partial f}{\partial x} = xe^{x^2+y^2} + 2xy$ $\xrightarrow[\text{wrt } x]{\text{integrate}}$ $\frac{1}{2}e^{x^2+y^2} + x^2y + h(y)$

$\frac{\partial f}{\partial y} = ye^{x^2+y^2} + x^2$ $\xrightarrow[\text{wrt } y]{\text{int.}}$ $\frac{1}{2}e^{x^2+y^2} + x^2y + g(x)$

So $f(x,y) = \frac{1}{2}e^{x^2+y^2} + x^2y$

(b) Find $\int_{\vec{c}} \vec{G}\cdot d\vec{s}$ along the path around the triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$



By F.T. of line integrals:

$$\int_{\vec{c}} \vec{G}\cdot d\vec{s} = \int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(c(\vec{a}))$$

(The path is pcws. cont. so we can apply the theorem)

moreover, $\vec{b} = \vec{a} = (0,0)$. So $\int_{\vec{c}} \vec{G}\cdot d\vec{s} = 0$

Let C be the curve $x^2+y^2=1$ lying in the plane $z=1$.

Let $\vec{F} = (z-y)\vec{i} + y\vec{k}$.

Calculate $\int_C \vec{F}\cdot d\vec{s}$ (you may use $\int_0^{2\pi} \sin^2 t\, dt = \pi$)

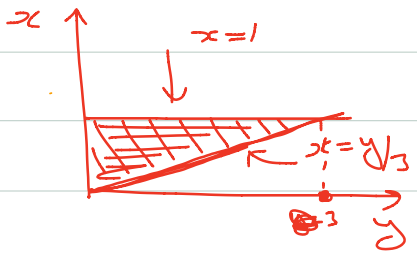
Let's parametrize: $\vec{c}(t) = (\cos t, \sin t, 1)$, $0 \le t \le 2\pi$

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} F(x(t), y(t), z(t)) \cdot \vec{c}'(t)\, dt$$

$$= \int_0^{2\pi} (1-\sin t, 0, \sin t)\cdot(-\sin t, \cos t, 0)\,dt = \int_0^{2\pi} -\sin t + \sin^2 t\,dt$$

$$= 0 + \int_0^{2\pi} \sin^2 t\,dt = \pi$$

Evaluate $\int_0^3 \int_{y/3}^1 e^{x^2}\,dx\,dy$ by changing the order of integration.



so the region can be described by $0 \le y \le 3x$, $0 \le x \le 1$

$$\Rightarrow \int_0^1 \int_0^{3x} e^{x^2}\,dy\,dx = \int_0^1 3x e^{x^2}\,dx = \frac{3}{2} e^{x^2}\Big|_0^1 = \frac{3}{2}(e-1)$$

Let D be the region bounded by the lines

$$x+y=0,\ x+y=2,\ x-y=0,\ x-y=2$$

Evaluate $\iint_D (x+y)\, e^{x^2-y^2}\,dx\,dy$

using C. of V.: $u = x+y$, $v = x-y \Rightarrow x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

$$\left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \right| = \left| -\frac{1}{2} \right|$$

$$\Rightarrow \int_0^2 \int_0^2 u e^{uv} \left(\tfrac{1}{2}\right) du\,dv = \tfrac{1}{2} \int_0^2 \int_0^2 u e^{uv}\,dv\,du = \tfrac{1}{2} \int_0^2 e^{uv}\Big|_{v=0}^{v=2} du$$

$$= \left(\tfrac{1}{2}\right) \int_0^2 (e^{2u} - 1)\,du = \frac{1}{2}(e^4 - 5)$$

Find an equation for the tg. plane of $f(x,y) = ye^{xy}$ at $(1,0)$

$f(1,0) = 0$

$\frac{\partial f}{\partial x}\Big|_{(1,0)} = y^2 e^{xy}\Big|_{(1,0)} = 0$

$\frac{\partial f}{\partial y}\Big|_{(1,0)} = e^{xy} + xye^{xy}\Big|_{(1,0)} = 1$

$\Rightarrow$ tg. plane: $z = 0 + 0(x-1) + 1(y-0)$

$\Leftrightarrow z = y$

Write the 2nd order taylor formula for f at $(1,0)$

$\frac{\partial^2 f}{\partial x^2}\Big|_{(1,0)} = y^3 e^{xy}\Big|_{(1,0)} = 0$, $\frac{\partial^2 f}{\partial x \partial y}\Big|_{(1,0)} = \frac{\partial^2 f}{\partial y \partial x}\Big|_{(1,0)} = xy^2 e^{xy} + 2ye^{xy}\Big|_{(1,0)} = 0$

$\frac{\partial^2 f}{\partial y^2}\Big|_{(1,0)} = xe^{xy} + xe^{xy} + x^2 ye^{xy}\Big|_{(1,0)} = 2$

Taylor's formula: $f(x,y) = y + \begin{bmatrix} x-1 & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix} + R_2(x,y)$

$= y + 2y^2 + R_2$

$\lim_{(x,y)\to(1,0)} \left(R_2 / \|(x,y)-(1,0)\|^2 \right) = 0$